universe+

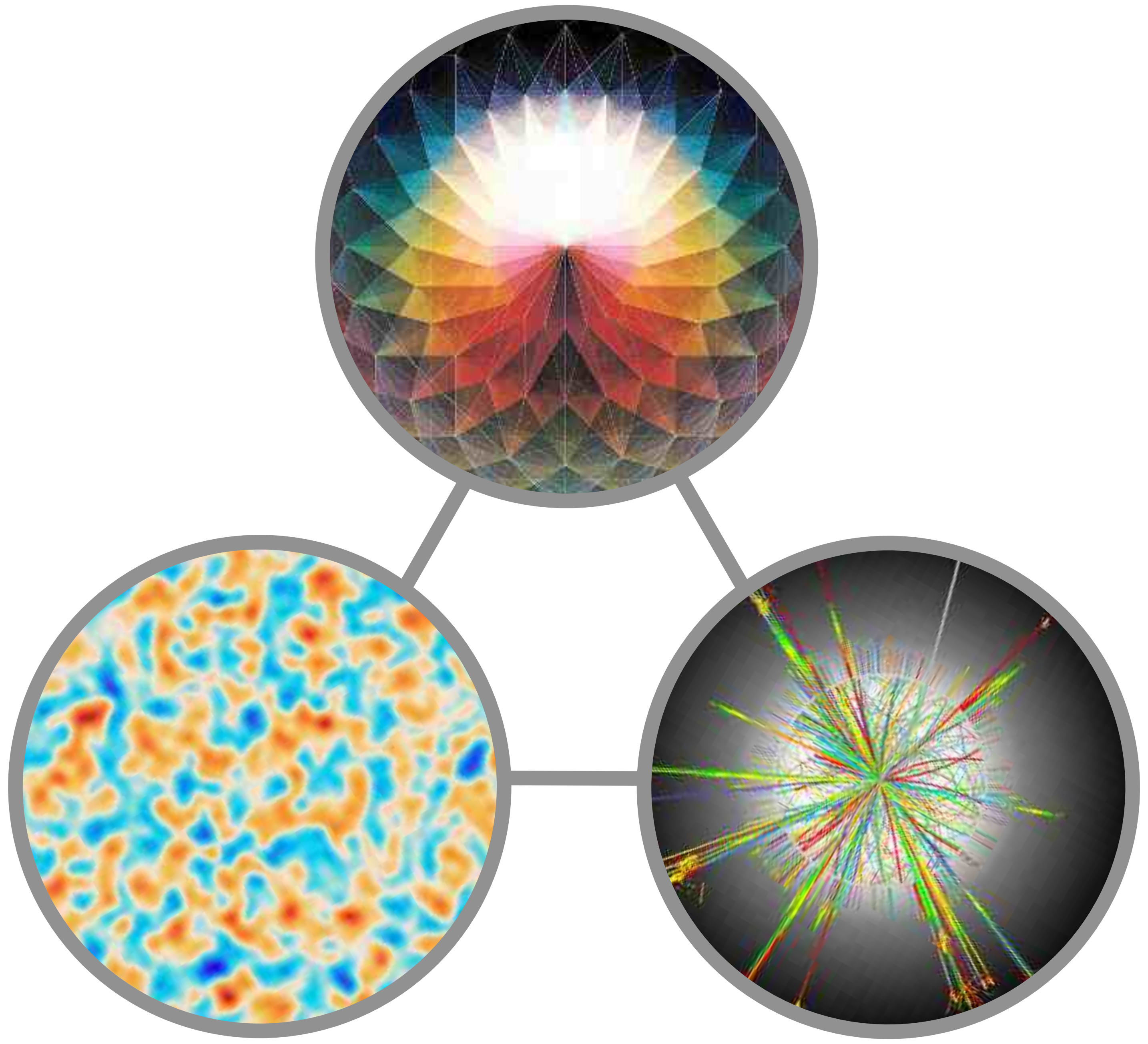
Positive Geometry in Particle Physics and Cosmology

Johannes M. Henn (Max Planck Institute for Physics)

Physikalisches Kolloquium, Universität Heidelberg, May 9, 2025

What is Positive Geometry?

Positive Geometry



Cosmology

Particle Physics

Hidden mathematical structures have been discovered in scattering amplitudes and in cosmological correlators.

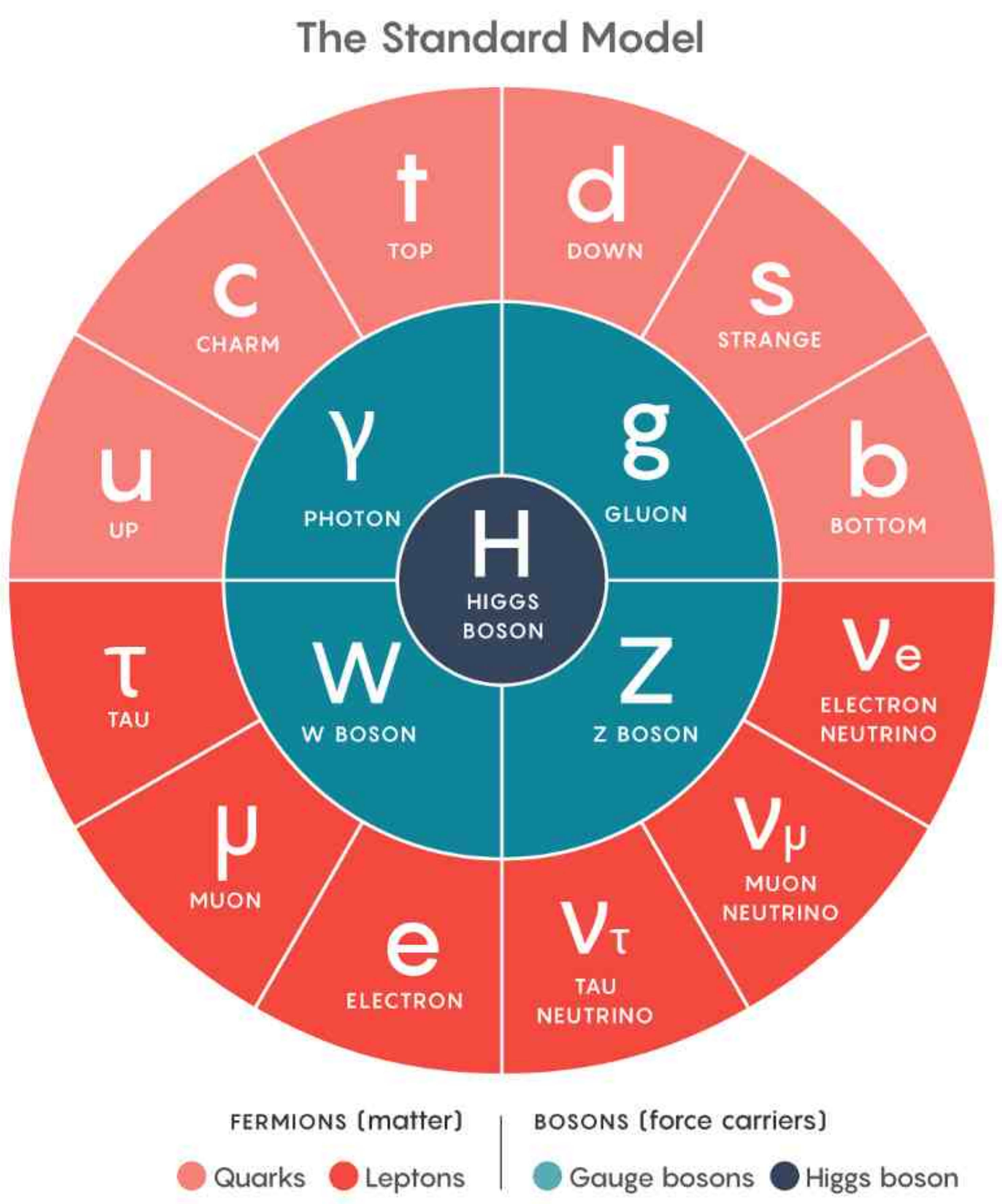
Outline

- Cast of characters:
  Scattering amplitudes, Cosmological Correlators
- What is Positive Geometry?
- Examples of discoveries and open questions

Standard Model of Particle Physics

The Standard Model of Particle Physics successfully describes elementary particles and their interactions, including the Higgs particle found at the Large Hadron Collider (Nobel Prize 2012).

Many fundamental open questions remain. In the decades to come, more data will be collected that serves to test the theoretical description, to determine fundamental constants of Nature, and to glean hints for a more complete theory.



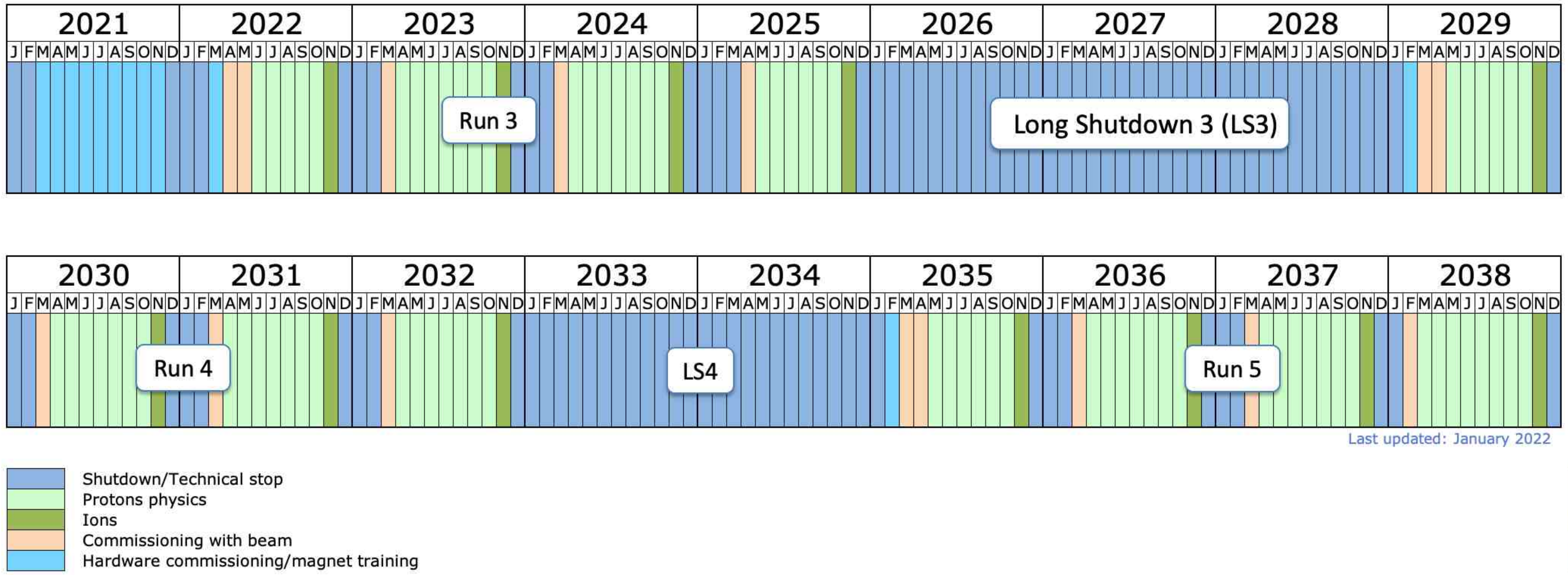
Ongoing experimental efforts at CERN, and long-term planning by the collider physics community

LHC long term schedule

https://lhc-commissioning.web.cern.ch/schedule/LHC-long-term.htm

Longer term LHC schedule

In January 2022, the schedule was updated with long shutdown 3 (LS3) to start in 2026 and to last for 3 years.



LHC and Injectors out to 2038

HL-LHC performance

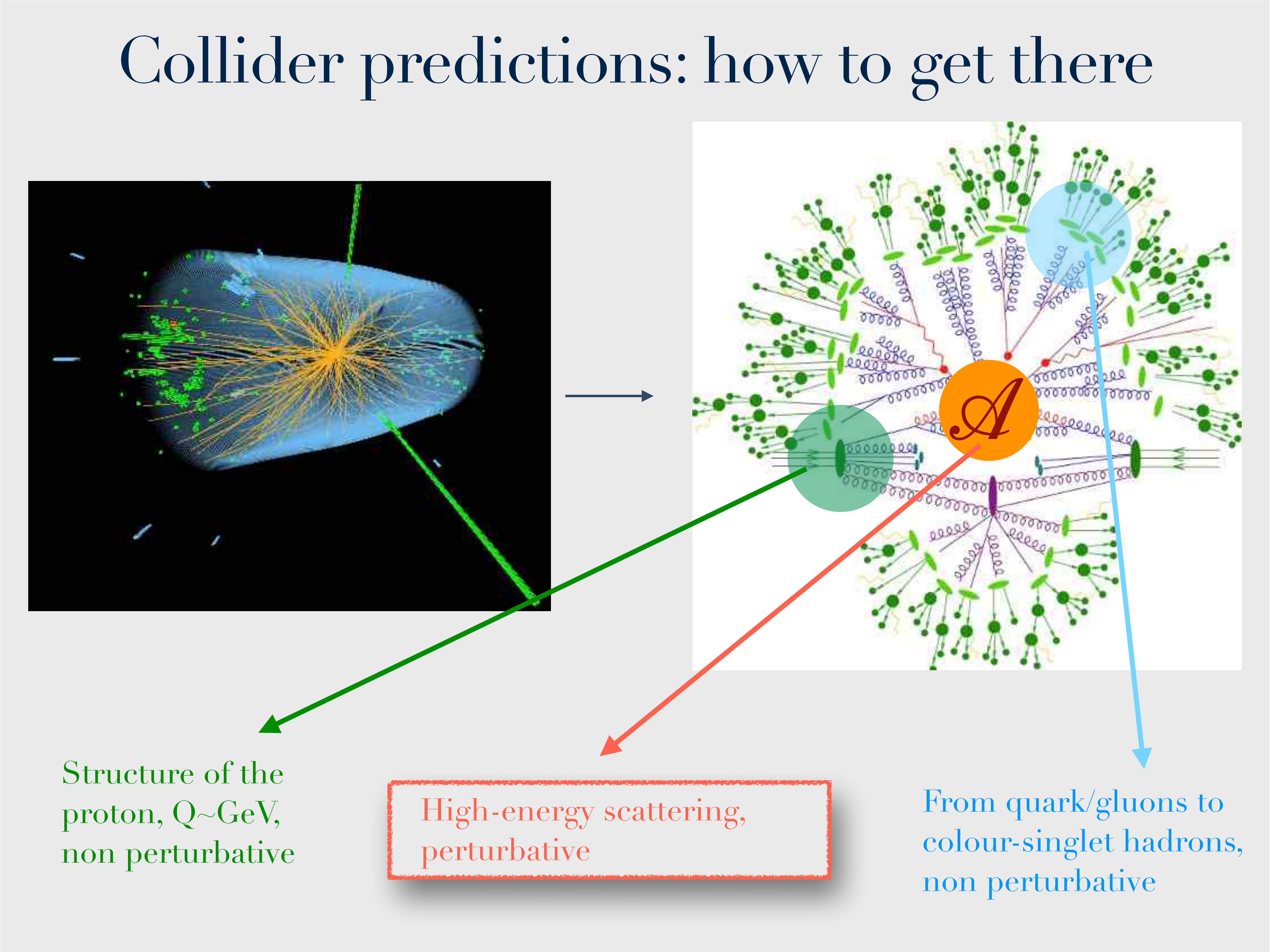
Canonical plots etc here

Run 3

Integrated luminosity targets

More details at LHC Performance Workshop 2022

Connecting theory and experiment

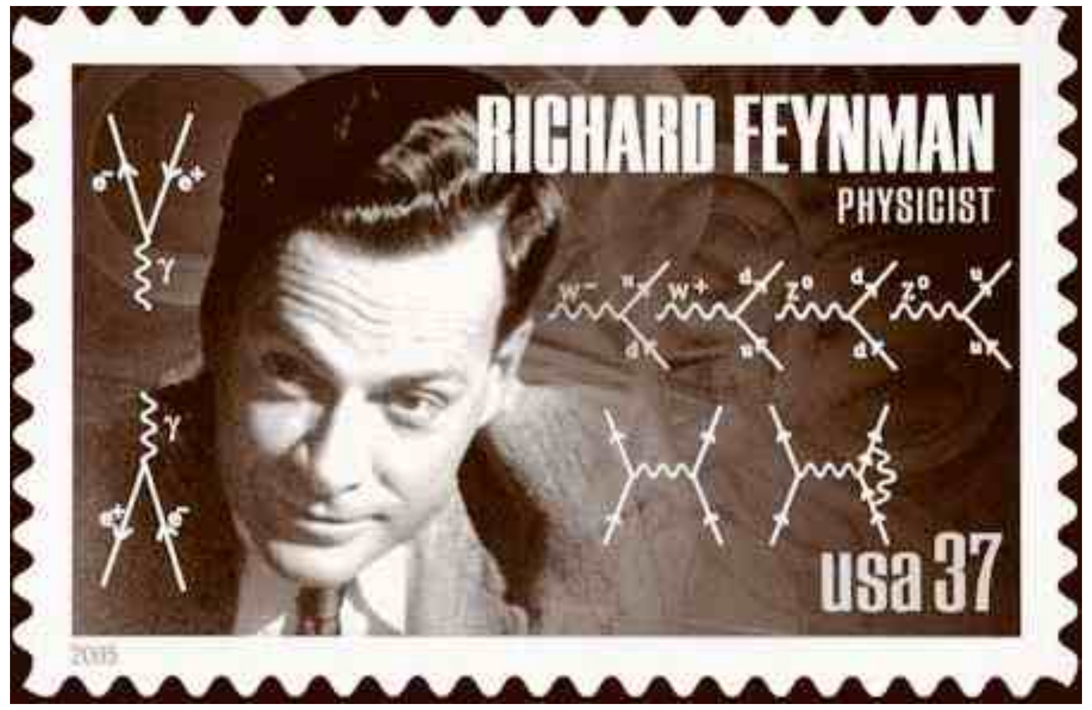


From Fabrizio Caola's talk at Amplitudes 2023, CERN]

Scattering amplitudes are key ingredients to the theoretical description of collider experiments.

Scattering amplitudes tell us about the probability of outcomes of particle interactions

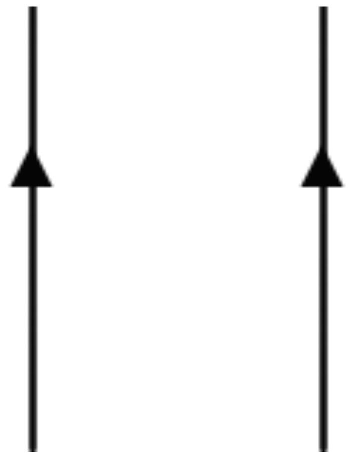
Feynman’s diagrams describe the interaction process. Each diagram stands for a mathematical formula.



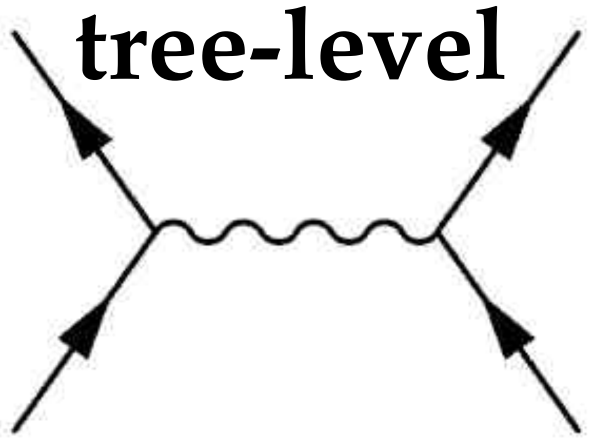
Perturbative expansion:

$$A(ee \to ee) = 1 + q^2 \cdot A_0 + q^4 \cdot A_1 + q^6 \cdot A_2 + \dots$$

‘NNLO’

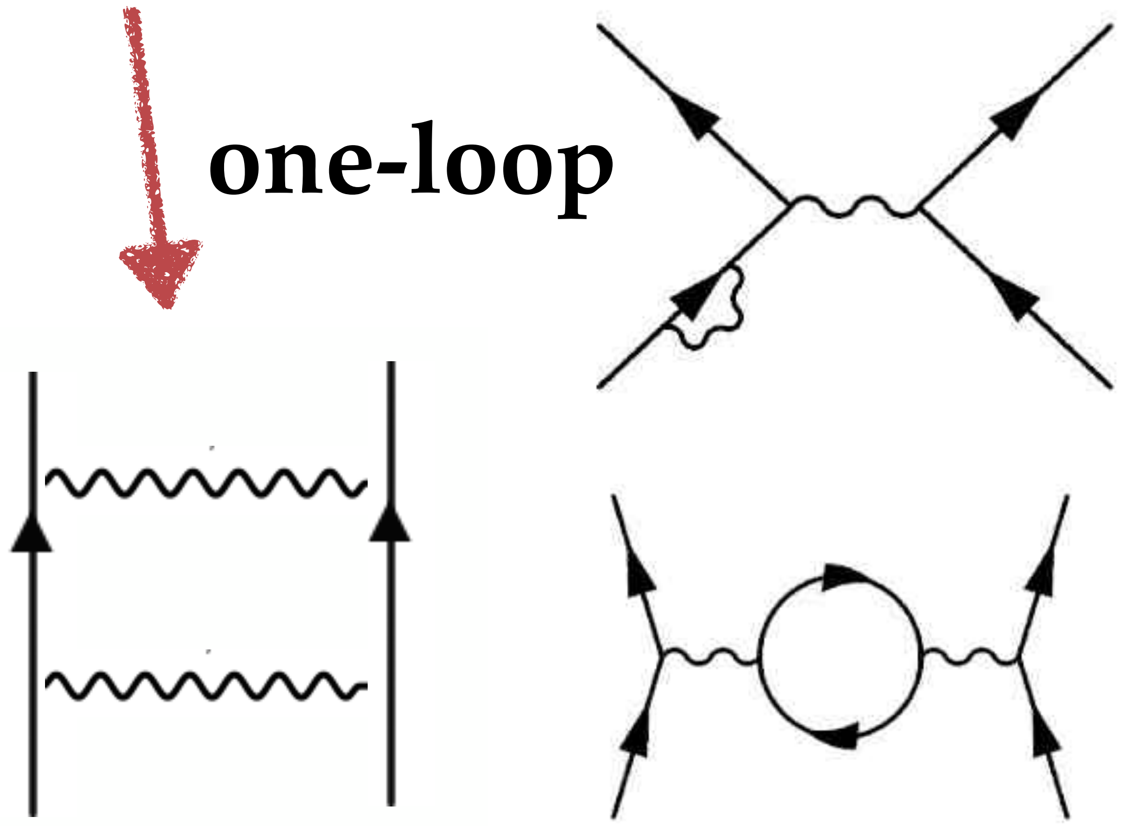


tree-level



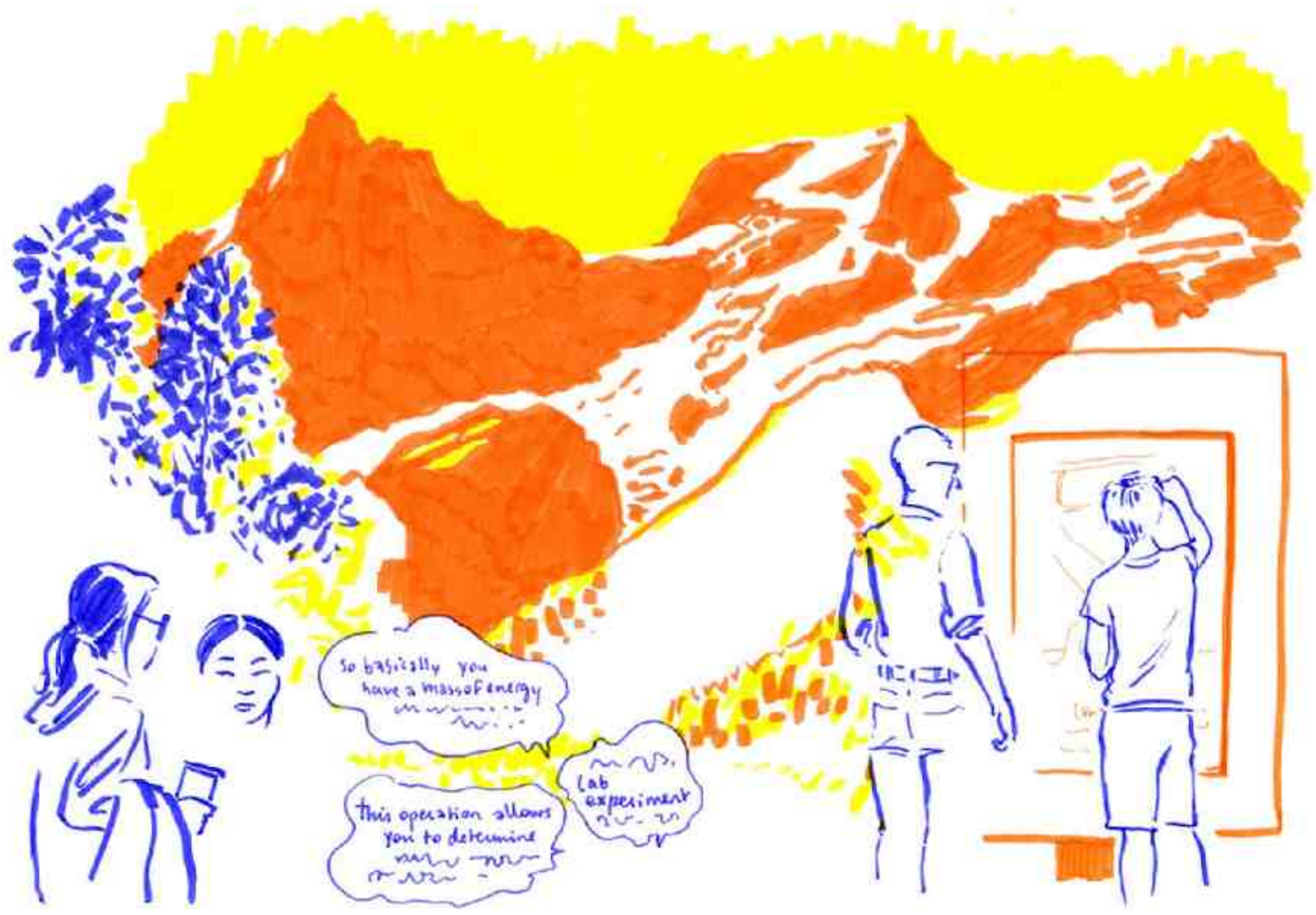
‘leading order’ (LO)

one-loop



‘next-to-leading order (NLO)’

‘Les Houches wishlist’ gives an idea of what is needed from an experimental viewpoint



NNLO QCD and NLO EW Les Houches Wishlist

Wishlist part 1 - Higgs (V=W,Z)

Process	known	desired	motivation
H	d\sigma @ NNLO QCD d\sigma @ NLO EW finite quark mass effects @ NLO	d\sigma @ NNNLO QCD + NLO EW MC@NNLO finite quark mass effects @ NNLO	H branching ratios and couplings
H+j	d\sigma @ NNLO QCD (g only) d\sigma @ NLO EW	d\sigma @ NNLO QCD + NLO EW finite quark mass effects @ NLO	H p_T
H+2j	\sigma_tot(VBF) @ NNLO(DIS) QCD d\sigma(gg) @ NLO QCD d\sigma(VBF) @ NLO EW	d\sigma @ NNLO QCD + NLO EW	H couplings
H+V	d\sigma(V decays) @ NNLO QCD d\sigma @ NLO EW	with H→bb @ same accuracy	H couplings
t\bar tH	d\sigma(stable tops) @ NLO QCD	d\sigma(NWA top decays) @ NLO QCD + NLO EW	top Yukawa coupling
HH	d\sigma @ LO QCD finite quark mass effects d\sigma @ NLO QCD large m_t limit	d\sigma @ NLO QCD finite quark mass effects d\sigma @ NNLO QCD	Higgs self coupling

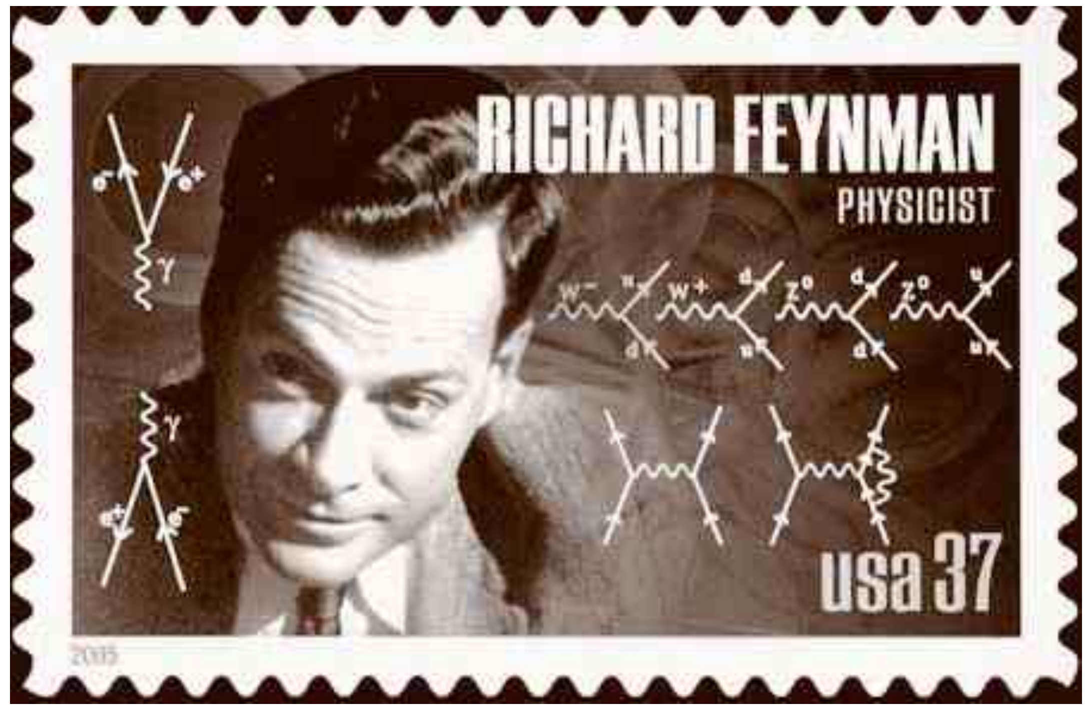
The previous ‘NLO’ Les Houches wishlist is by now essentially obsolete, thanks to tremendous theory advances. Today, calculations at NNLO and even beyond are needed to match the experimental precision.

How to compute scattering amplitudes?

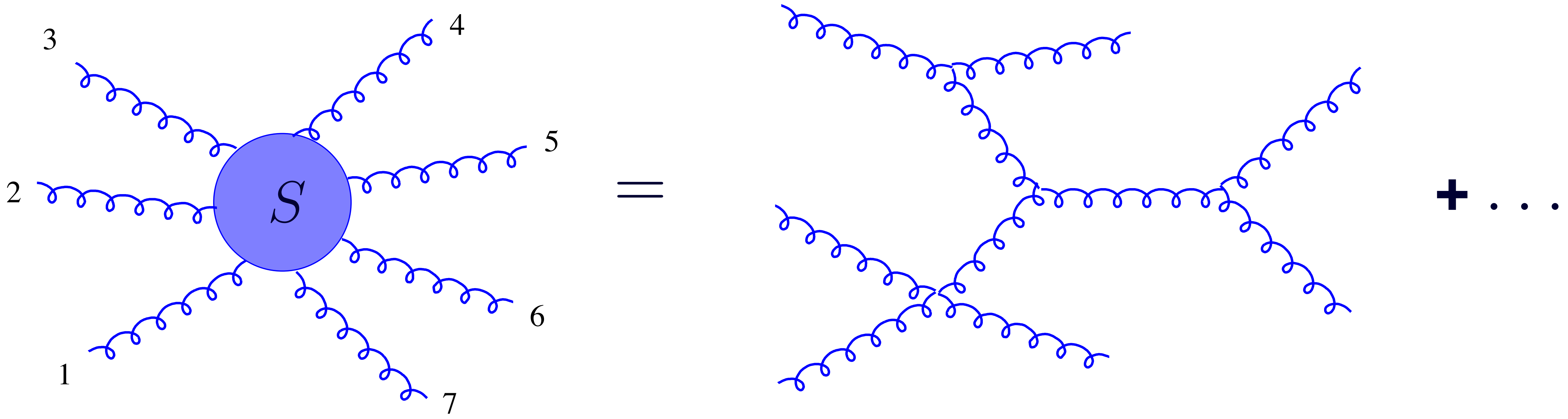
We know how to do this in principle:

(1) draw all Feynman diagrams

(2) compute them



Often not so simple in practice! E.g. for tree-level gluon amplitudes:



number of external gluons	4	5	6	7	8	9	10
number of diagrams	4	25	220	2485	34300	559405	10525900

State-of-the art in the early 1980’s

Most complicated process: $gg \to ggg$ at leading order

Brute force calculation
24 pages of result

$(k_1 \cdot k_4)(\epsilon_2 \cdot k_1)(\epsilon_1 \cdot \epsilon_3)(\epsilon_4 \cdot \epsilon_5)$

Perhaps not every question has a simple answer?

Hidden simplicity in scattering amplitudes

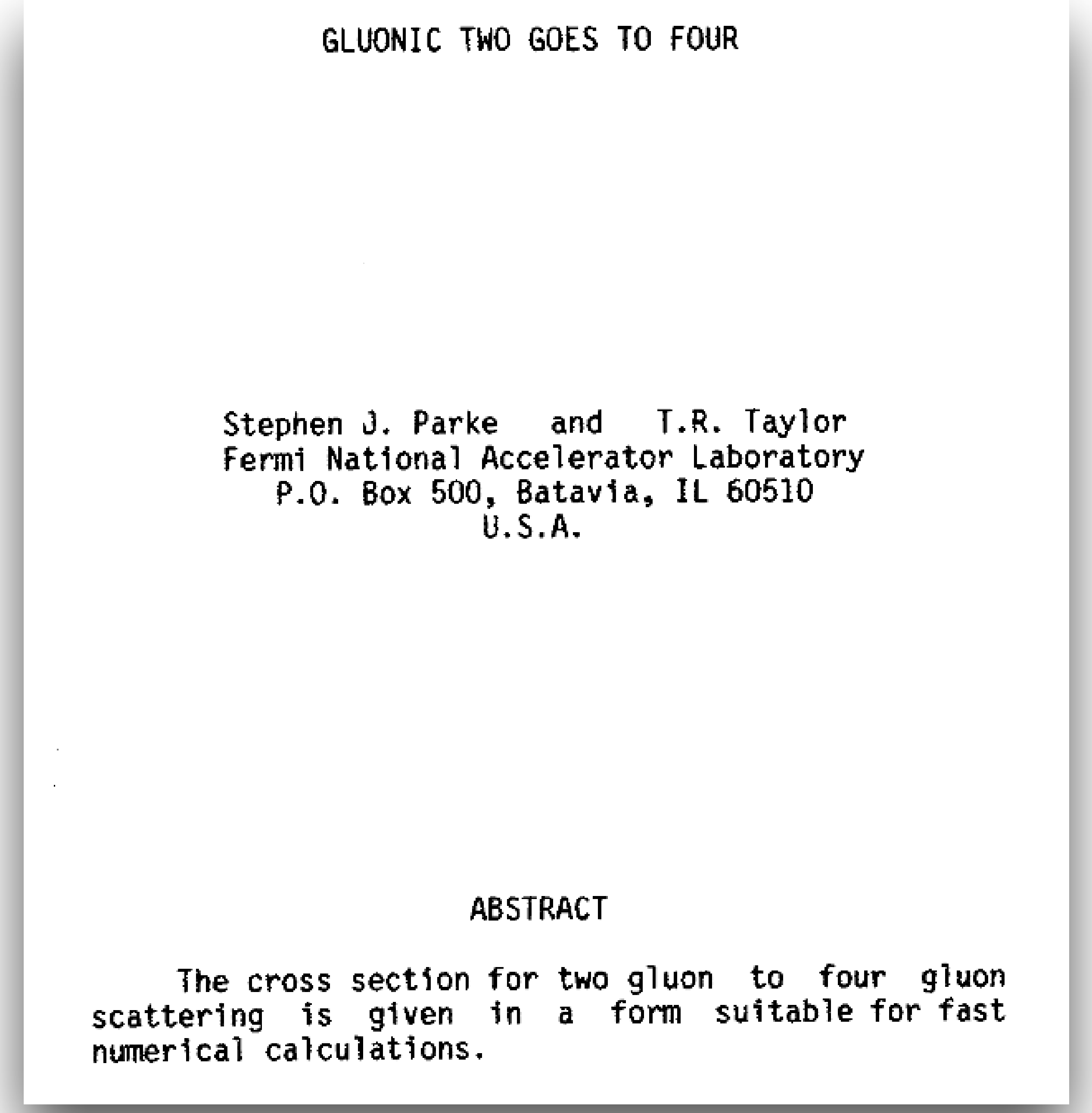
In 1983, the Superconducting Super Collider had been approved. Theorists were getting ready to compute cross sections for all the gluons to be produced at 40 TeV.

Parke, Taylor (1985)

Process $gg \to gggg$

220 Feynman diagrams, 100 pages of calculations

Paper with 14 pages of result



GLUONIC TWO GOES TO FOUR

Stephen J. Parke and T.R. Taylor
Fermi National Accelerator Laboratory
P.O. Box 500, Batavia, IL 60510
U.S.A.

ABSTRACT

The cross section for two gluon to four gluon scattering is given in a form suitable for fast numerical calculations.

Hidden simplicity in scattering amplitudes

> Our result has succesfully passed both these numerical checks.
>
> Details of the calculation, together with a full exposition of our techniques, will be given in a forthcoming article. Furthermore, we hope to obtain a simple analytic form for the answer, making our result not only an experimentalist's, but also a theorist's delight.

Parke, Taylor (1985)

Within a year, they realised

$$|A_6|^2 \sim \frac{(p_1 \cdot p_2)^3}{(p_2 \cdot p_3)(p_3 \cdot p_4)(p_4 \cdot p_5)(p_5 \cdot p_6)(p_6 \cdot p_1)}$$

Final answer much simpler than the intermediate expressions.

Are there better ways of obtaining the result?

‚Philosophy' of the amplitudes community

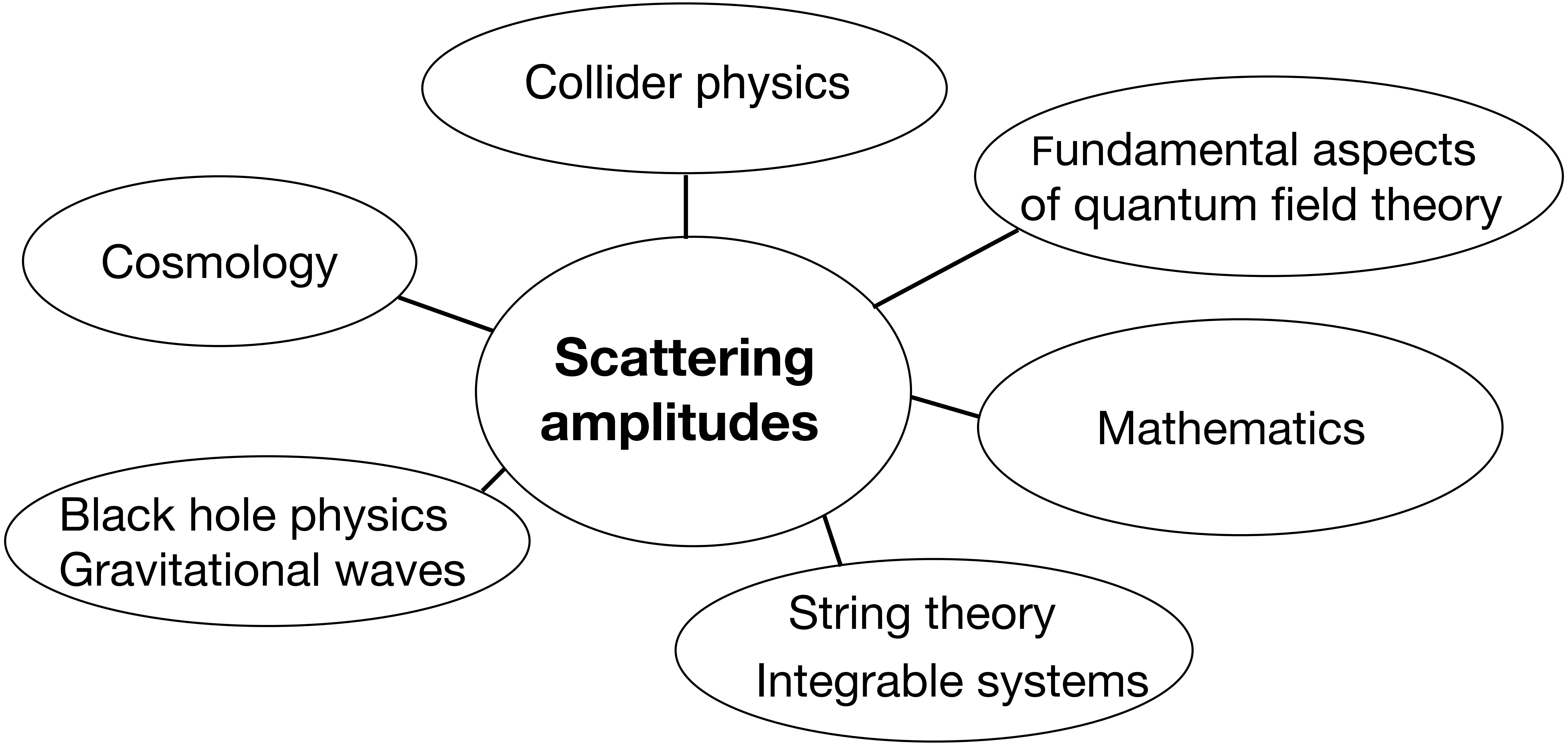
Why are the final results so simple?

Are there other ways for obtaining them?

How can we calculate amplitudes relevant for the LHC?

Calculated amplitudes serve as useful ‚mathematical data'. Insights into their structure can lead to better ways of obtaining them.

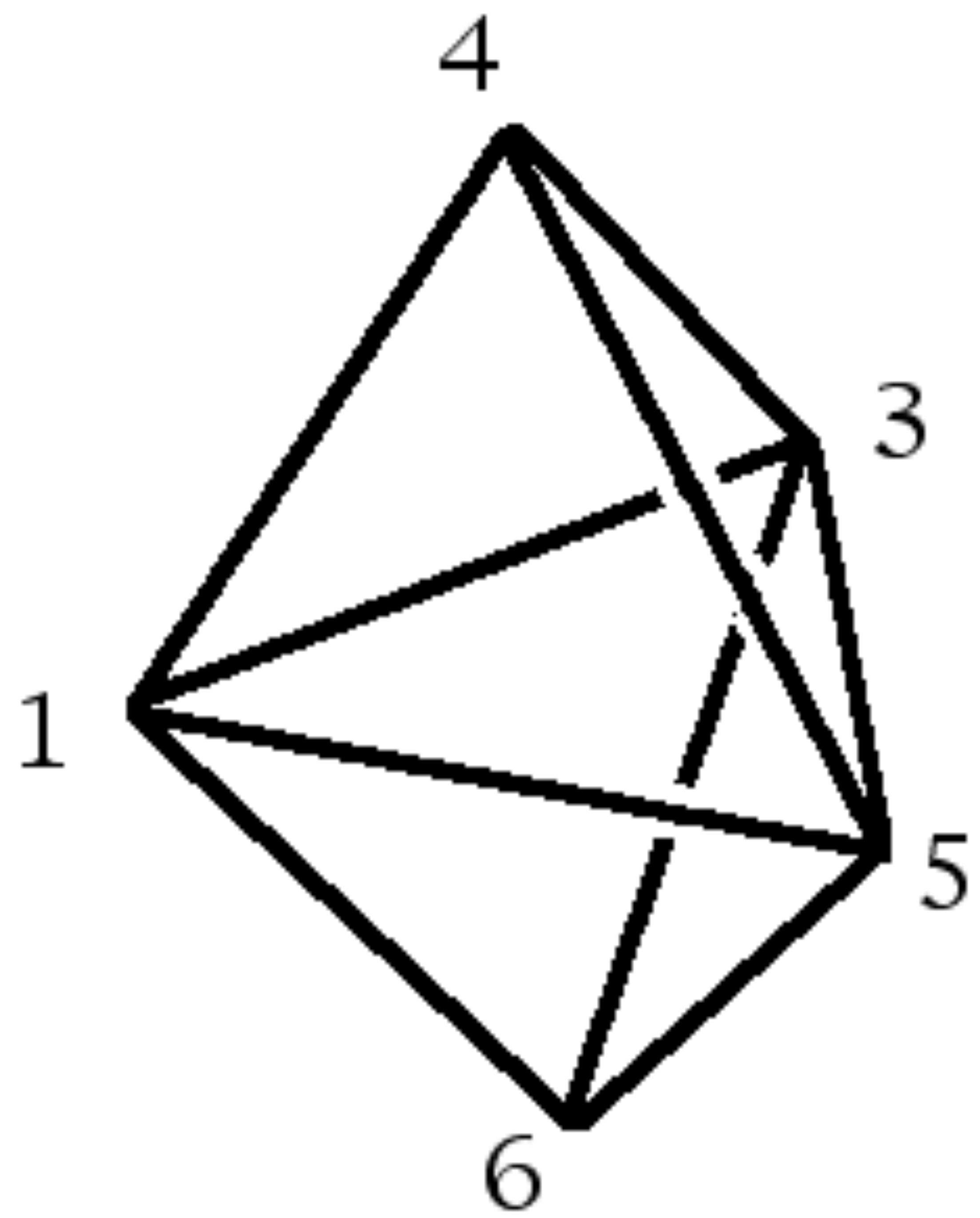
An interconnected research field



Leads to beneficial cross-fertilisation between the different areas.

Scattering amplitudes as volumes

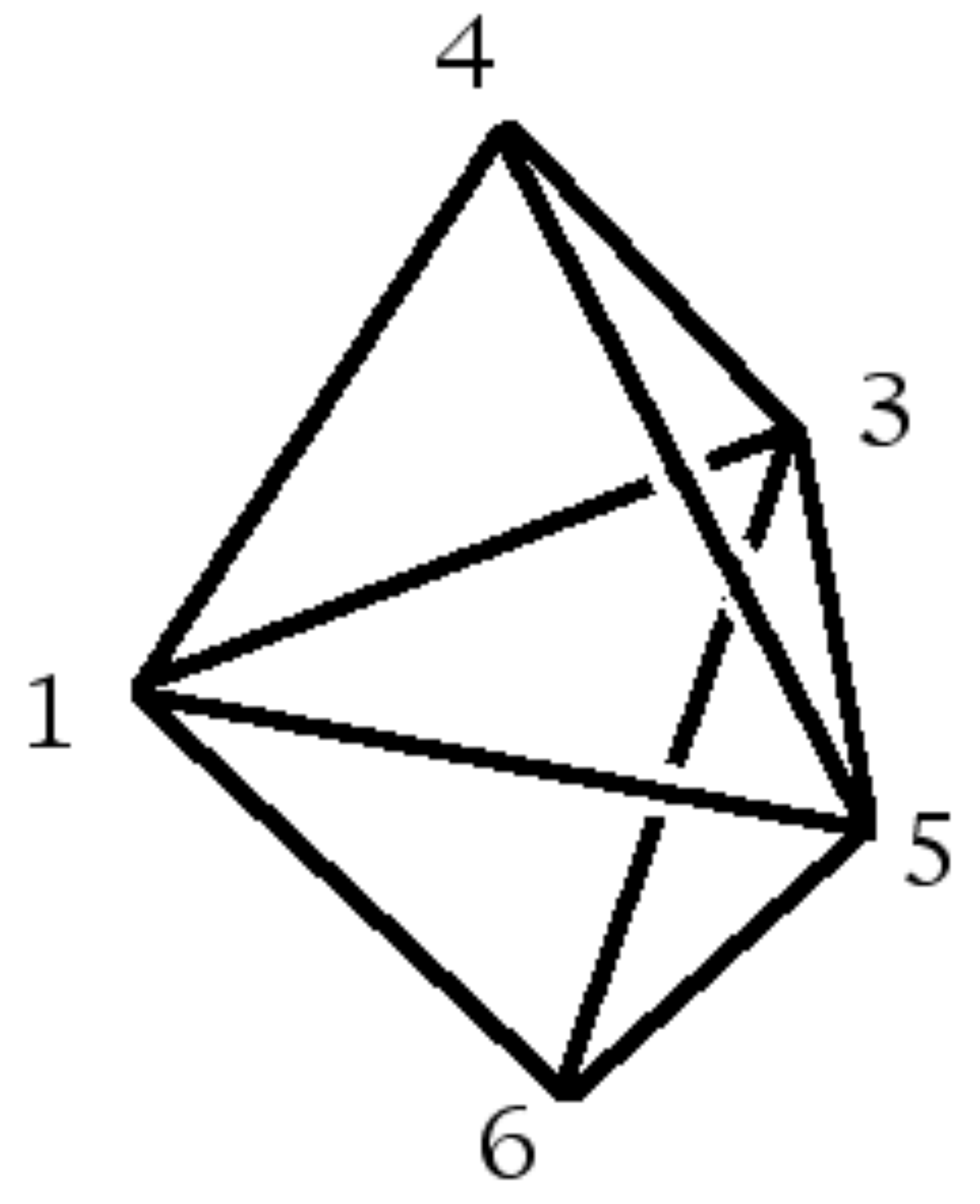
In 2009, Andrew Hodges found that certain tree-level Yang-Mills helicity amplitudes are given by the volume of a certain polytope, whose vertices are determined by the kinematics that describes the six-particle scattering.



[figure from Hodges (2009)]

New properties thanks to geometric definition

- Completely new perspective on the answer, different from summing Feynman diagrams
- Triangulations of volume can be used for novel decompositions of the amplitude
- Facet structure of the polytope informs about physical properties, such as singularities and discontinuities
- Definition is linked to positivity. Positivity properties of scattering amplitudes follow
- ….



[figure from Hodges (2009)]

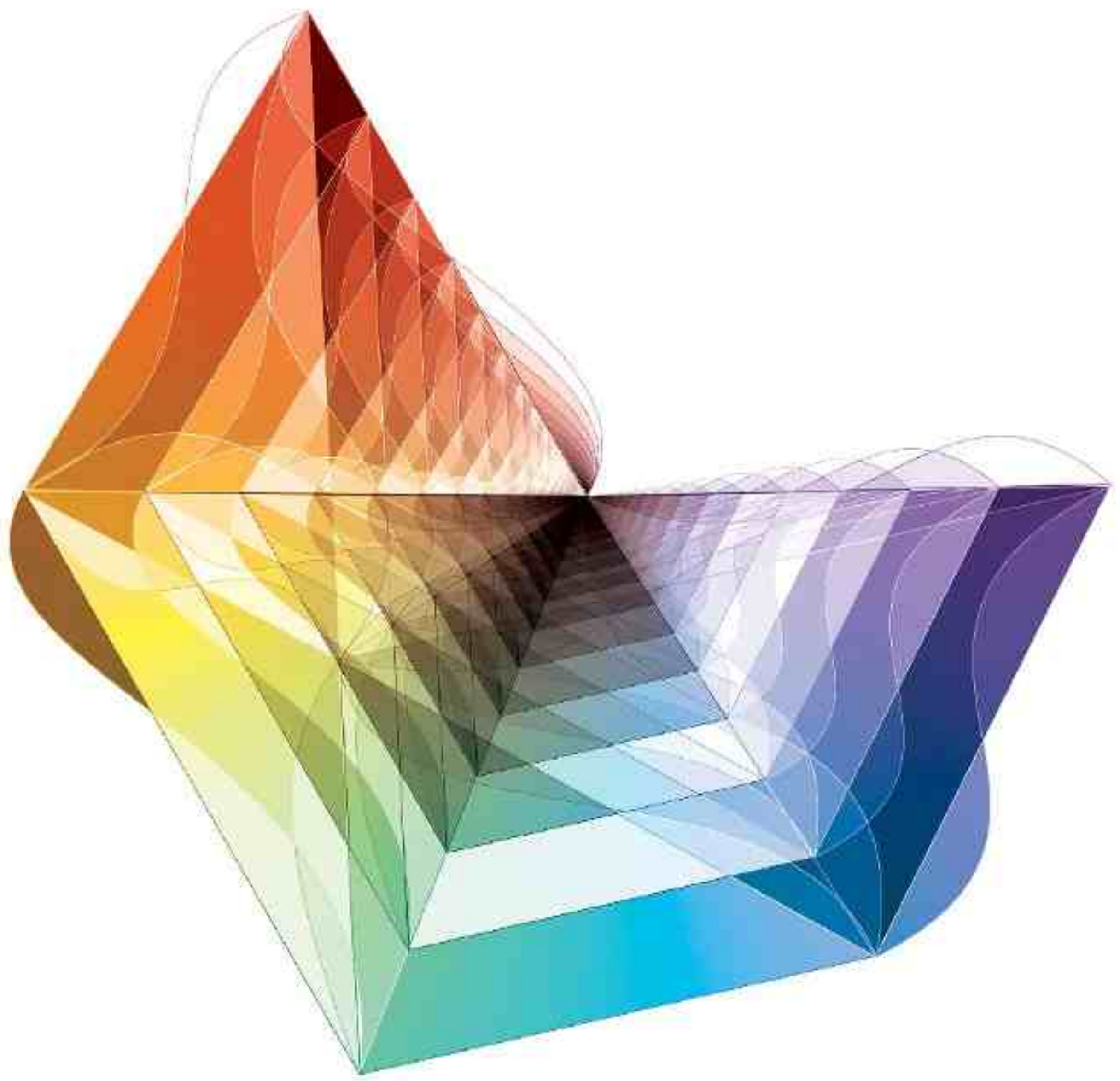
But is it just a coincidence for certain tree-level scattering amplitudes?

Amplituhedron

In 2013, Nima Arkani-Hamed and Jaroslav Trnka generalized this to all helicity amplitudes, and to any loop order in perturbation theory, for the loop integrand.

They achieved this in maximally supersymmetric Yang-Mills theory, a toy model for a four-dimensional quantum field theory.

In this case, the geometry is more complicated than a polytope, it is ‚curvy'.

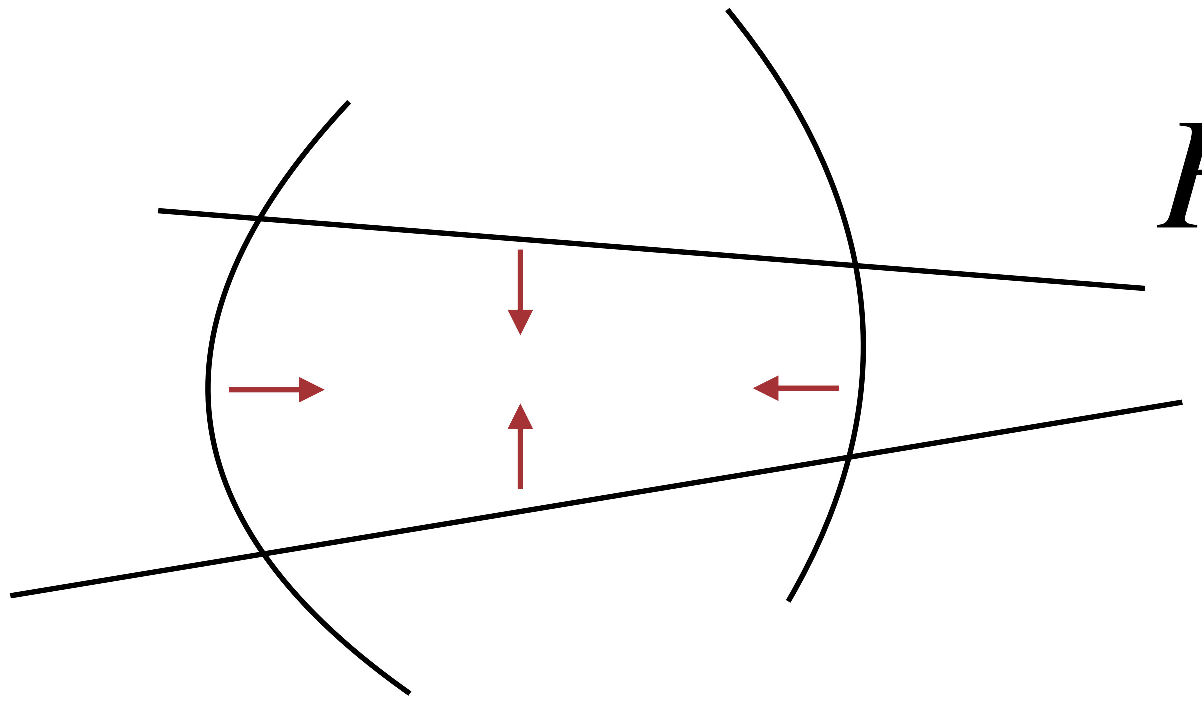


'A Jewel at the Heart of Quantum Physics'
— Natalie Wolchover, Quanta Magazine

Positive Geometry

Space of dimension D, defined by polynomial inequalities.

Kinematics $\longrightarrow$ parameters x_i .



$$F(x_i) \geq 0$$

Canonical form with logarithmic singularities at the boundaries.

$$\omega_D \xrightarrow[x=0]{} \frac{dx}{x}\omega_{D-1}$$

Recursive definition.

<u>Example:</u> line segment.

$$F_1(x) = x - x_1 > 0$$
$$F_2(x) = x_2 - x > 0$$

$x = x_1$ $\qquad$ $x = x_2$

$$\Omega = \frac{dx\,(x_1 - x_2)}{(x - x_1)(x - x_2)}$$

Novel perspective: from geometry to functions

1. Geometry defined by set of inequalities
(in physical or auxiliary space x, y)

$\longrightarrow$ which geometries are relevant?

$\downarrow$

2. Canonical differential form:
(rational) Feynman *integrand* $\omega(x, y)$

$\longrightarrow$ how to construct the form in practice?

$\downarrow$

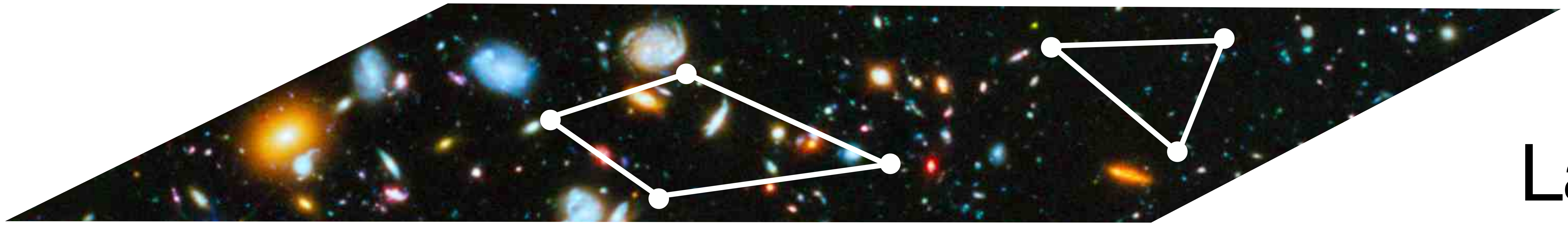
3. Feynman *integral:*
special function $\quad f(x) = \int \omega(x, y)$

$\longrightarrow$ how to perform the integrations?

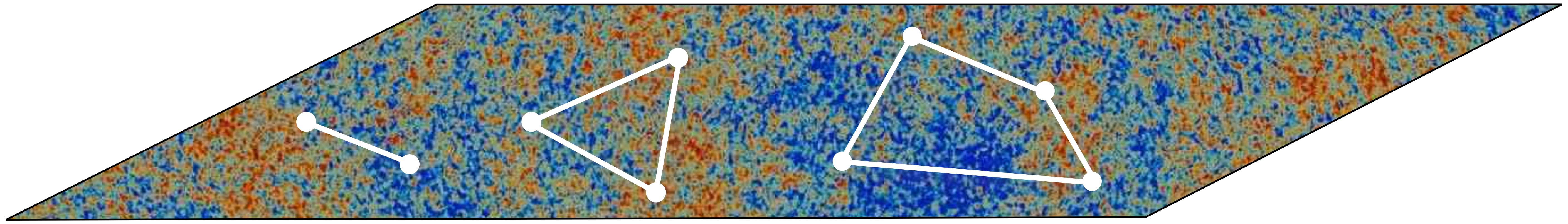
Examples of known or conjectured positive geometries

- *Amplituhedron:* describes scattering amplitudes in maximally supersymmetric Yang-Mills theory. Tree-level version heavily studied by mathematicians.
- *Associahedron*: describes scattering amplitudes in bi-adjoint cubic scalar theory.
- *EFT-hedron:* gives bounds on effective field theories.
- *Cosmological polytopes / Cosmohedra:* describe inflationary correlation functions in cosmology.
- …

Cosmological Correlators



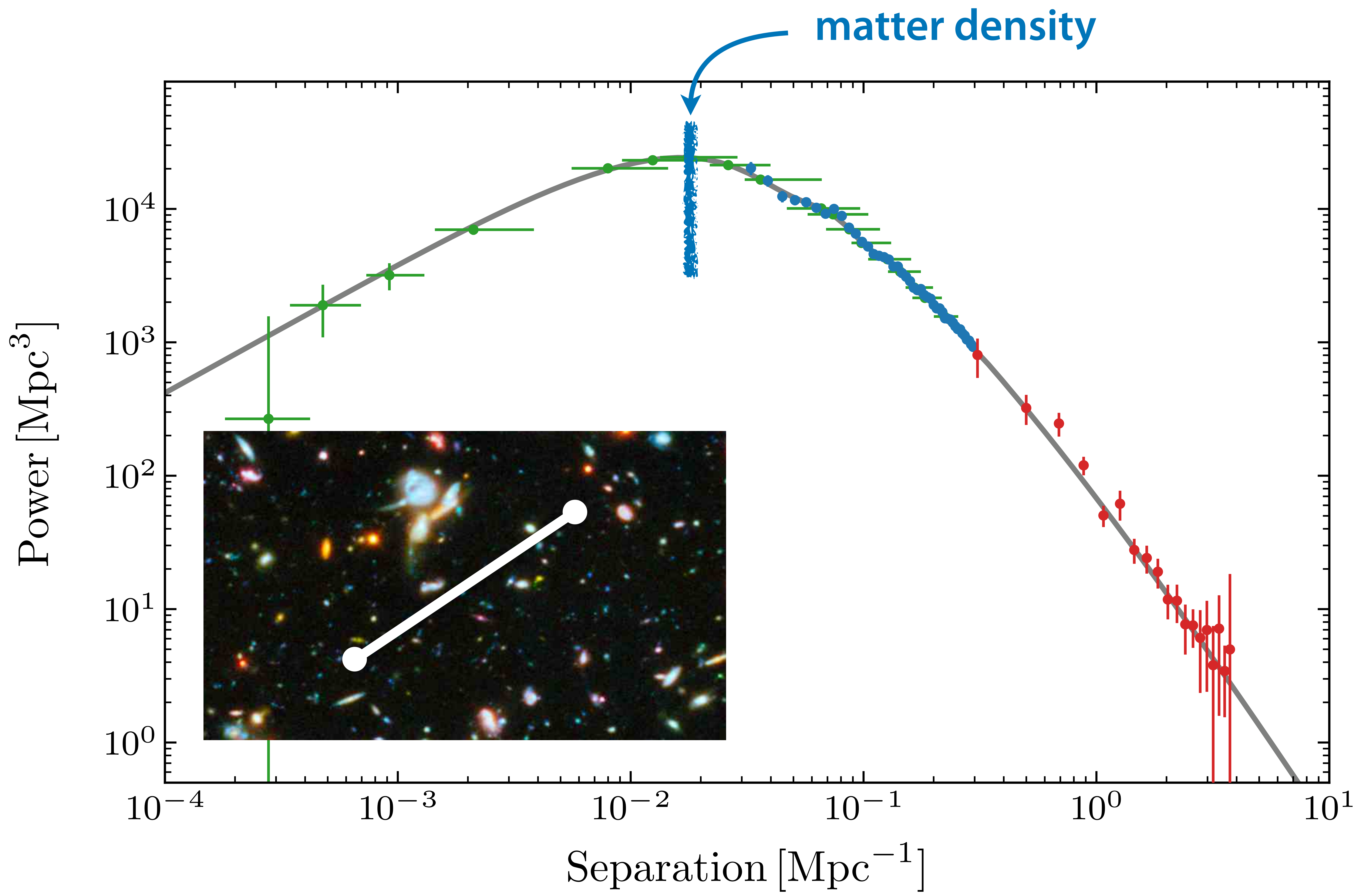
Large Scale Structure (LSS)



Cosmic Microwave Background (CMB)

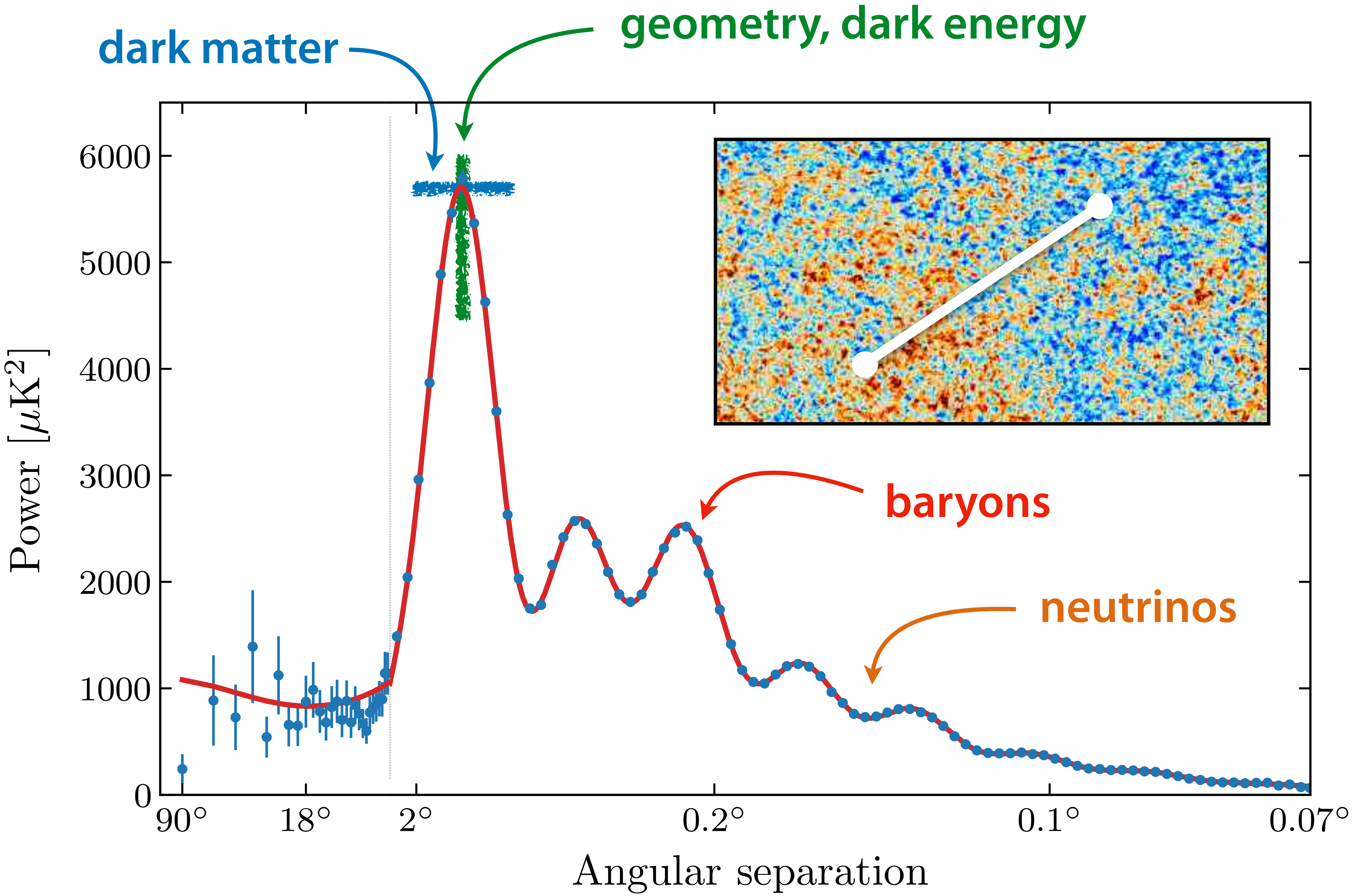
The main observables in Cosmology are correlation functions.

Correlations in the distribution of galaxies



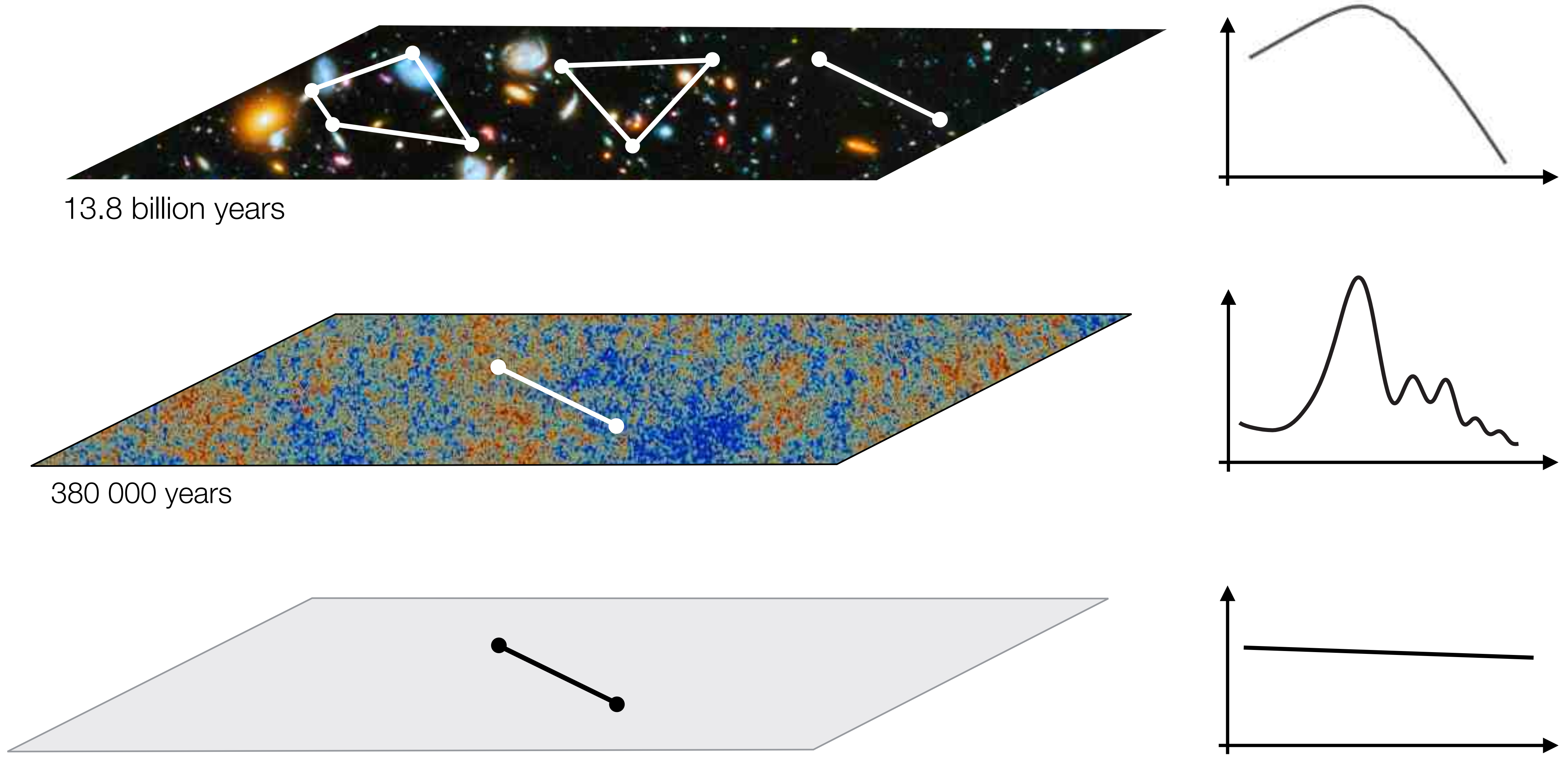
[Figure from Daniel Baumann's talk at Leipzig, 02/2024]

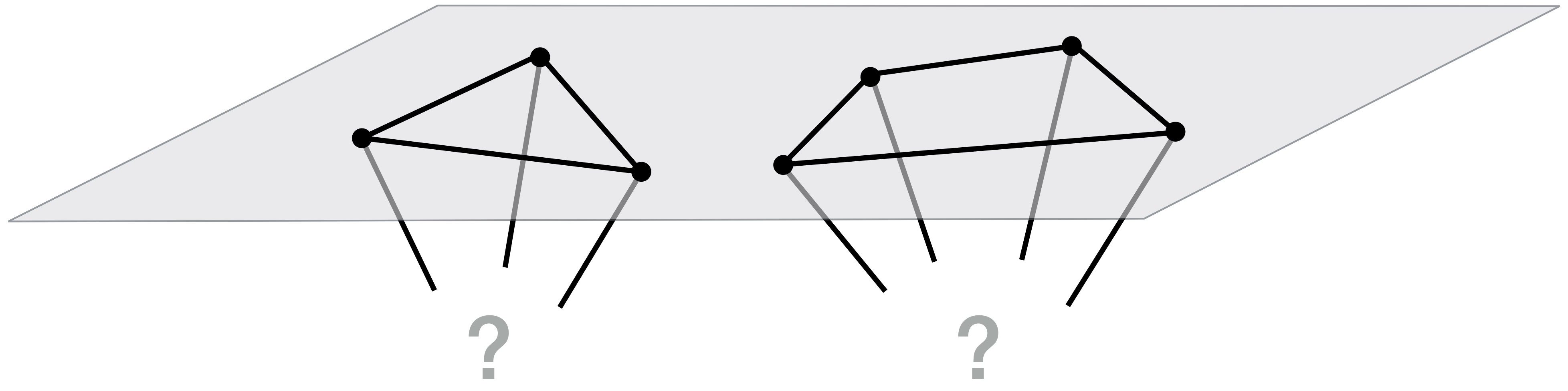
Correlations in the CMB temperature



[Figure from Daniel Baumann's talk at Leipzig, 02/2024]

Primordial Correlations





These correlations can be traced back to the `Hot Big Bang'.

We wish to learn about the physic before the Hot Big Bang. So far, mostly two-point correlations studied and related to the observational data.

[Figures from Daniel Baumann's talk at Leipzig, 02/2024]

Recap Cosmological Correlators

In Cosmology, quantum correlations at the end of inflation seed the perturbations that, via the Standard Model of Cosmology, lead to features of the Large Scale Structure and the Cosmic Microwave Background that we observe today.

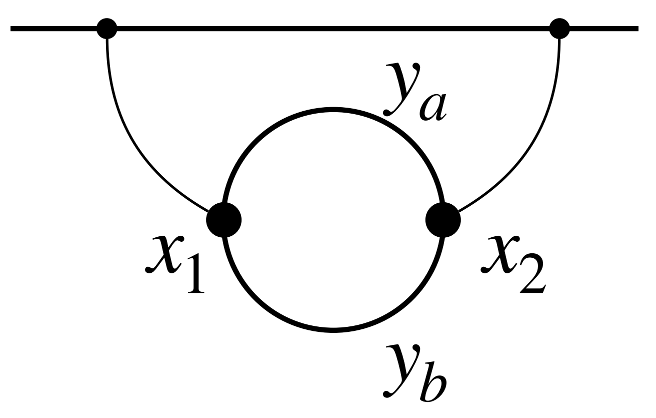
It is interesting to know them for different models of inflationary physics. For example, the presence of novel heavy particles may lead to specific features to look for.

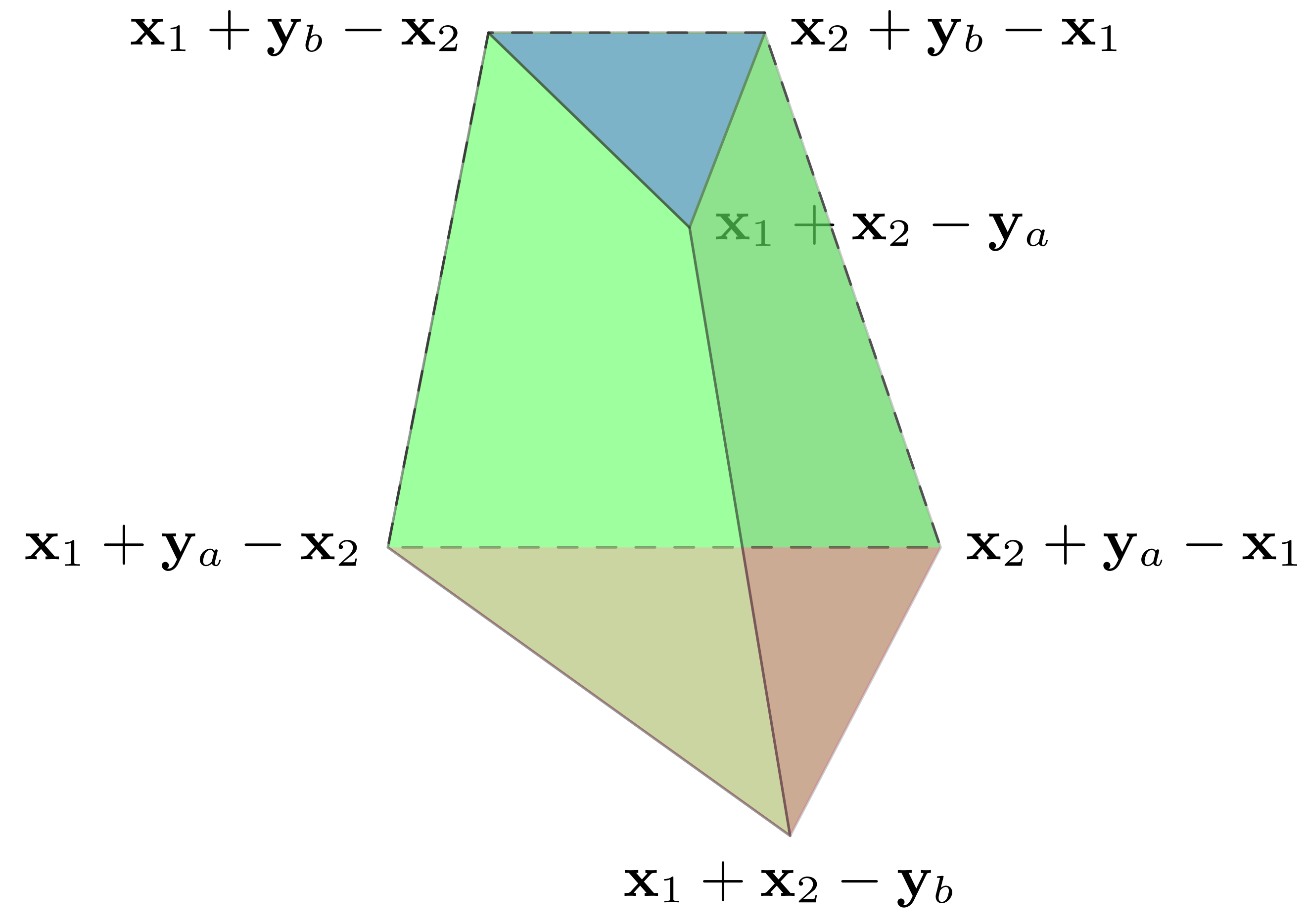
More complicated setting in curved space-time, but many ideas from Amplitudes apply!

Cosmological Polytopes

In 2017, Arkani-Hamed, Benincasa and Postnikov found cosmological polytopes for graphs in certain toy cosmological models.

Example: integrand for a two-point function with loop correction, for conformally coupled scalars in Friedmann–Lemaître–Robertson–Walker spacetime.





$$\psi_2^{(1)} = \frac{2(x_1+x_2+y_a+y_b)}{(x_1+x_2)(x_1+y_a+y_b)(x_2+y_a+y_b)(x_1+x_2+2y_a)(x_1+x_2+2y_b)},$$

[Arkani-Hamed, Benincasa, Postnikov (2017)]

From positive geometries to Positive Geometry

Positive geometries have been found that describe scattering amplitudes in various special quantum field theories.

They have also been observed in Feynman diagrams in cosmology.

Challenge: generalisations are needed to describe theories relevant to the *real word*, sich as quantum chromodynamics, or full cosmological correlators in realistic models.

Mathematics insights may provide invaluable guidance.

Can Positive Geometry describe particle scattering?

AMPLITUDES EXAMPLES

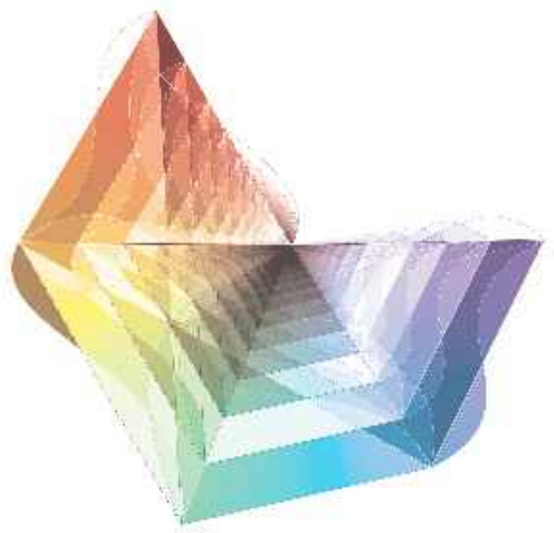
For amplitudes we have two prime examples:

Planar $\mathcal{N} = 4$ SYM theory

Amplituhedron

(Arkani-Hamed, JT, 2013)

tree-level & loop integrands



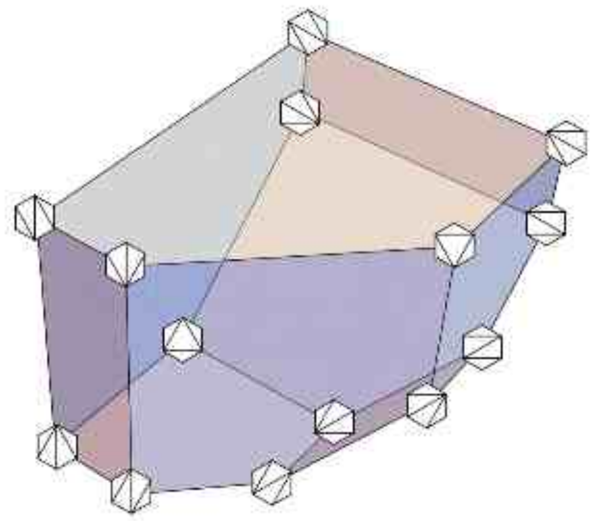
"Simplest QFT"
hidden symmetries

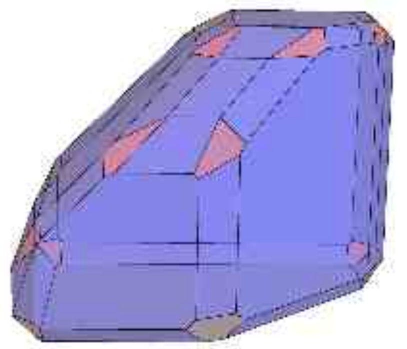
$\mathrm{Tr}(\phi^3)$/ bi-adjoint ϕ^3

ABHY Associahedron

(Arkani-Hamed, Bai, He, Yan, 2017)

tree-level amplitudes





Cosmohedron
for cosmological
correlators

(Arkani-Hamed, Figueiredo, Vazao, 2024)

Loop generalization:
Surfacehedron

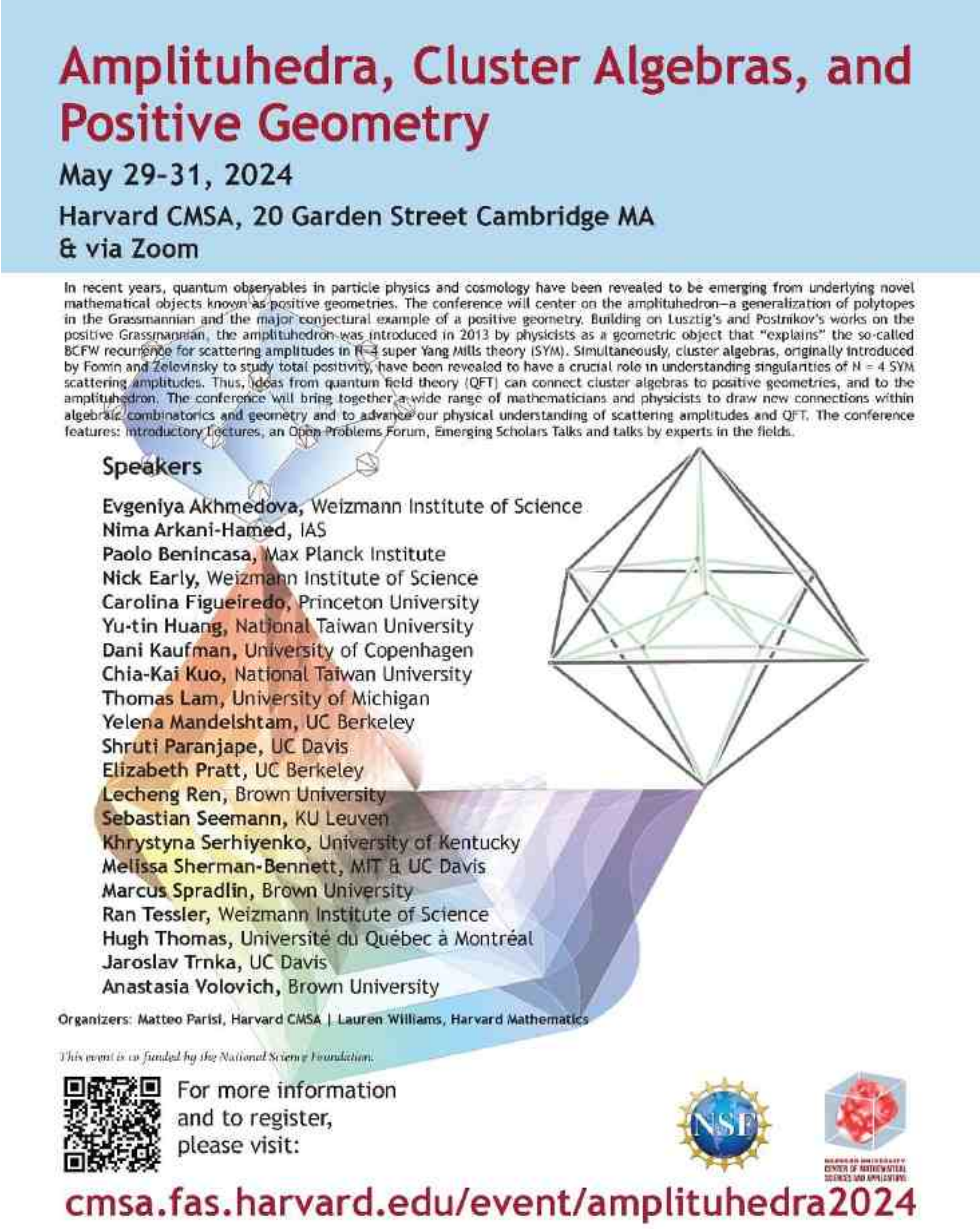
[from Jaroslav Trnka's talk on 10.3.2025, KITP]

Active field of research at the boundary of mathematics and physics

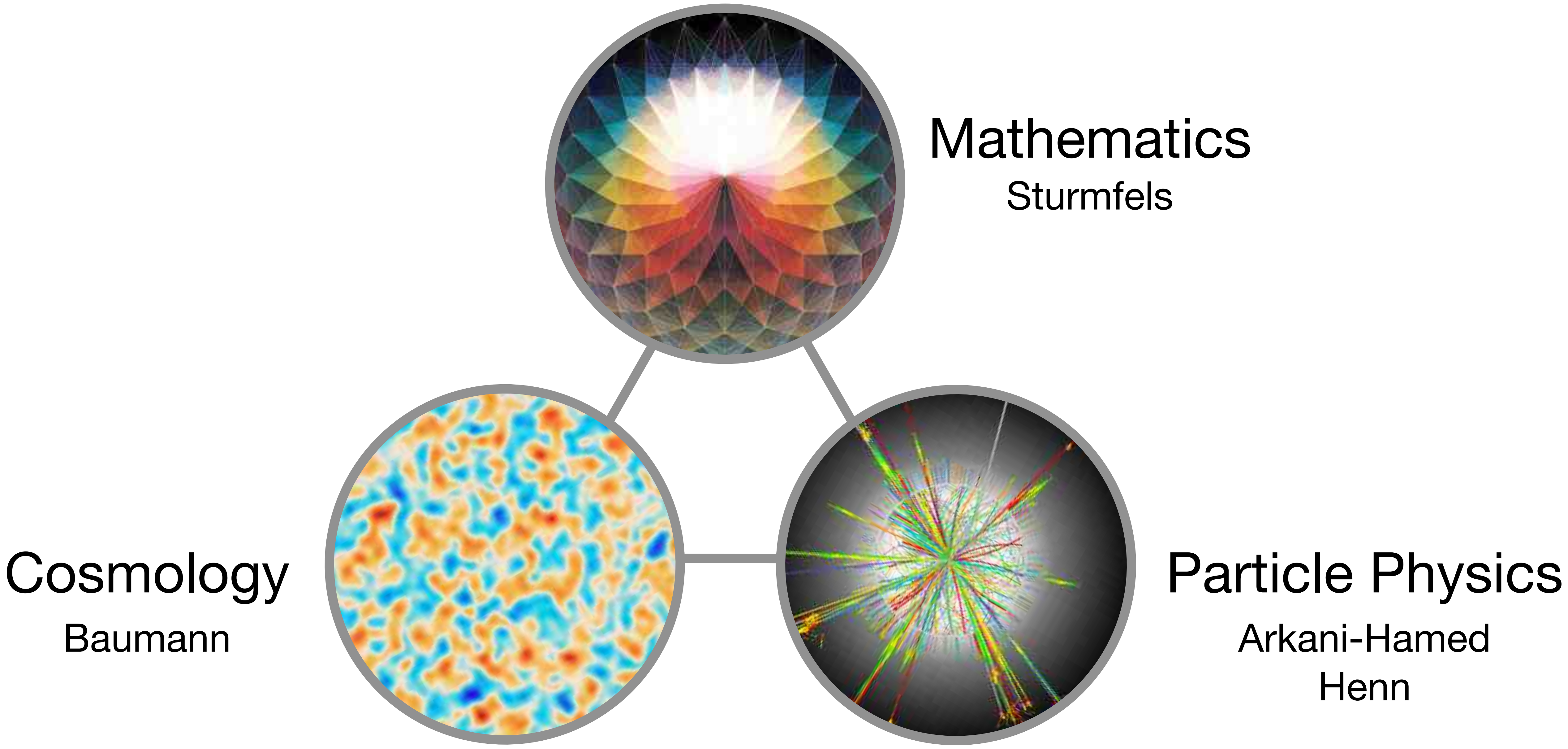
First example: Amplituhedron [Arkani-Hamed, Trnka, 2013]

Mathematics definition of positive geometries: [Arkani-Hamed, Bai, Lam, 2017]

Important work by many authors: Paolo Benincasa, Michael Borinsky, Lara Bossinger, Francis Brown, Taro V. Brown, Shounak De, Clément Dupont, Chaim Even-Zohar, Gabriele Dian, Nick Early, Livia Ferro, Carolina Figueiredo, Claudia Fevola, Haldeigh Frost, Pavel Galashin, Ross Glew, Song He, Enrico Herrmann, Paul Heslop, Yu-tin Huang, Aidan Herderschee, Ryota Kojima, , Hugh Thomas, Ran Tessler, Shruti Paranjape, Matteo Parisi, Andrzej Pokraka, Alexander Postnikov, Lizzie Pratt, Prashanth Raman, Anna-Laura Sattelberger, Melissa Sherman-Bennett, Akshay Yelleshpur Srikant, Bernd Sturmfels, Giulio Salvatori, Mark Spradlin, Jonah Stahlknecht, Tsvicqa Lakrec, Tomasz Lukowski, Cameron Langer, Fatima Mohammadi, Saiei-Jaeyeong Matsubara-Heo, Elia Mazzucchelli, Umut Oktem, Dimitri Pavlov, Francisco Vazão, Anastasia Volovich, Lauren Williams, Qinglin Yang, Minshan Zheng…



Positive Geometry in Particle Physics and Cosmology



In the ERC-funded UNIVERSE+ project (2024-2030), we seek a novel mathematical language to describe physical phenomena at all scales, building upon hints for new geometric structures that appeared both in particle physics and cosmology.

The research institutions and team

University of Amsterdam

Max Planck Institute for Mathematics
in the Sciences, Leipzig

Institute for Advanced Study, Princeton

Max Planck Institute for Physics, Munich

Positive Geometry in Particle Physics and Cosmology

Leipzig MPI-MiS workshop, February 2024

Upcoming special volume of Le Matematiche

WHAT IS POSITIVE GEOMETRY?

KRISTIAN RANESTAD - BERND STURMFELS - SIMON TELEN

This article serves as an introduction to the special volume on Positive Geometry in the journal Le Matematiche. We attempt to answer the question in the title by describing the origins and objects of positive geometry at this early stage of its development. We discuss the problems addressed in the volume and report on the progress. We also list some open challenges.

...

Positive geometry is an interdisciplinary subject inspired by fundamentally new ideas in particle physics and cosmology. Its origins lie in the observation that certain physical observables, called *scattering amplitudes* and *cosmological correlators*, can be recovered from the geometry of elegant mathematical objects, such as *amplituhedra* [7] and *cosmological polytopes* [5]. This has motivated physicists to search for such geometric structures in a range of quantum field theories. In positive geometry, we put this on a solid mathematical footing.

19 articles,

48 authors.

Introduction: ArXiv:2502.12815

List of 19 problems

A fruitful interplay of disciplines

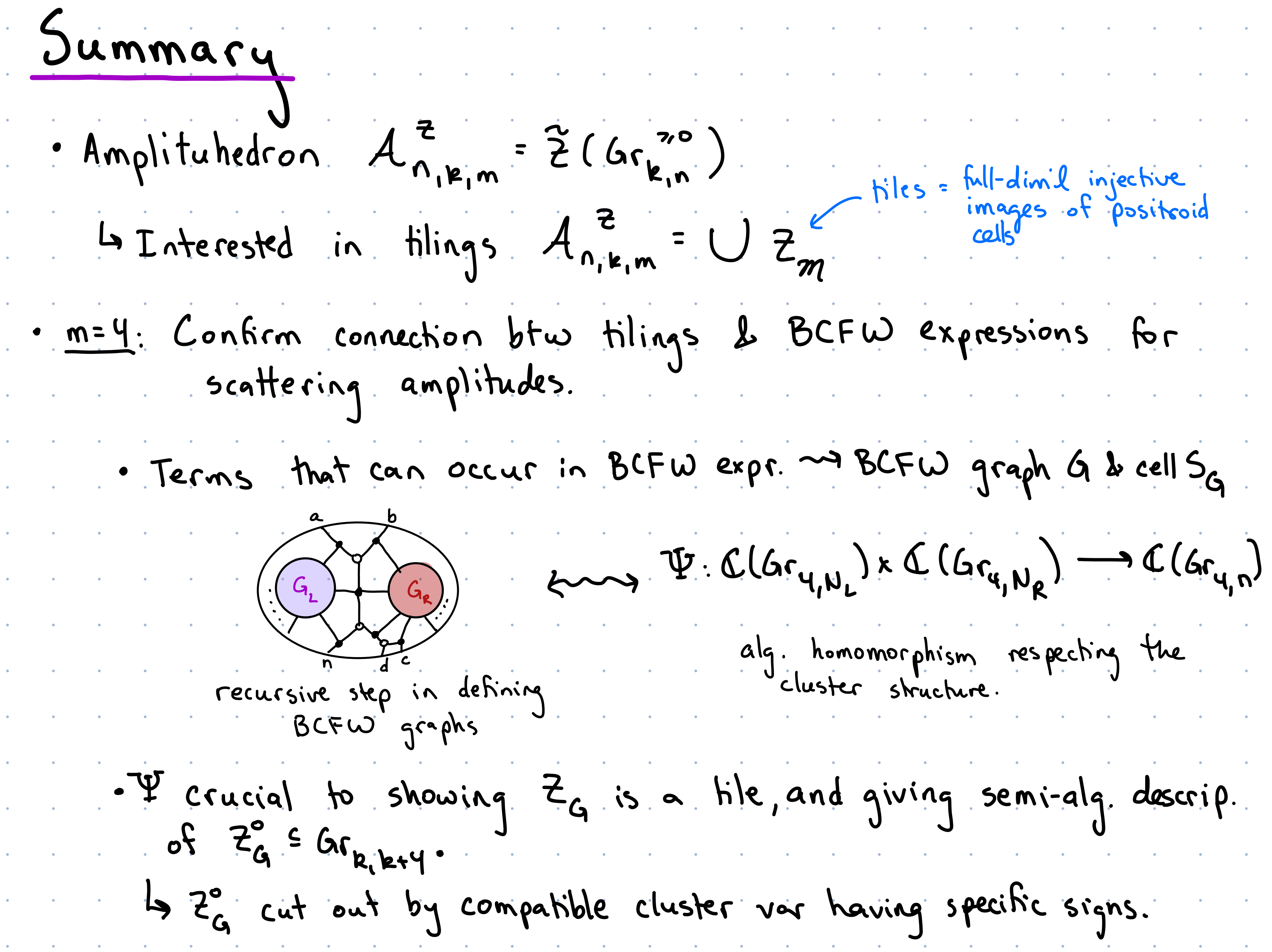
Positive Geometry is a developing field at the interface of mathematics and physics. This provides a novel set of fascinating ideas, conjectures to mathematics. Physics benefits from completely different mathematical viewpoints and approaches.

My goal is to find novel conceptual and practical methods, and to discover more positive geometries relevant to physics.

Examples of novel applications

- *Infrared and ultraviolet divergences:* Understand how the structure of divergences is implied by the geometry of the Feynman polytope.
- *Topical geometry methods* are useful for numerical integration and for understanding singular limits.
- *Novel geometry-based organization of QFT*, with the goal of understanding the final answers as certain volumes.
- …

Tilings of the tree-level Amplituhedron


Summary

• Amplituhedron $\mathcal{A}^{Z}_{n,k,m} = \tilde{Z}(Gr^{\geq 0}_{k,n})$

↳ Interested in tilings $\mathcal{A}^{Z}_{n,k,m} = \bigcup Z_{\mathfrak{m}}$

tiles = full-dim'l injective images of positroid cells

• m=4: Confirm connection btw tilings & BCFW expressions for scattering amplitudes.

• Terms that can occur in BCFW expr. ⇝ BCFW graph G & cell S_G

$\longleftrightarrow \Psi: \mathbb{C}(Gr_{4,N_L}) \times \mathbb{C}(Gr_{4,N_R}) \longrightarrow \mathbb{C}(Gr_{4,n})$

recursive step in defining BCFW graphs

alg. homomorphism respecting the cluster structure.

• Ψ crucial to showing Z_G is a tile, and giving semi-alg. descrip. of $Z^{\circ}_G \subseteq Gr_{k,k+4}$.

↳ Z°_G cut out by compatible cluster var having specific signs.

[from Melissa Sherman-Bennett (UC Davis)'s talk at Leipzig, 02/2024]

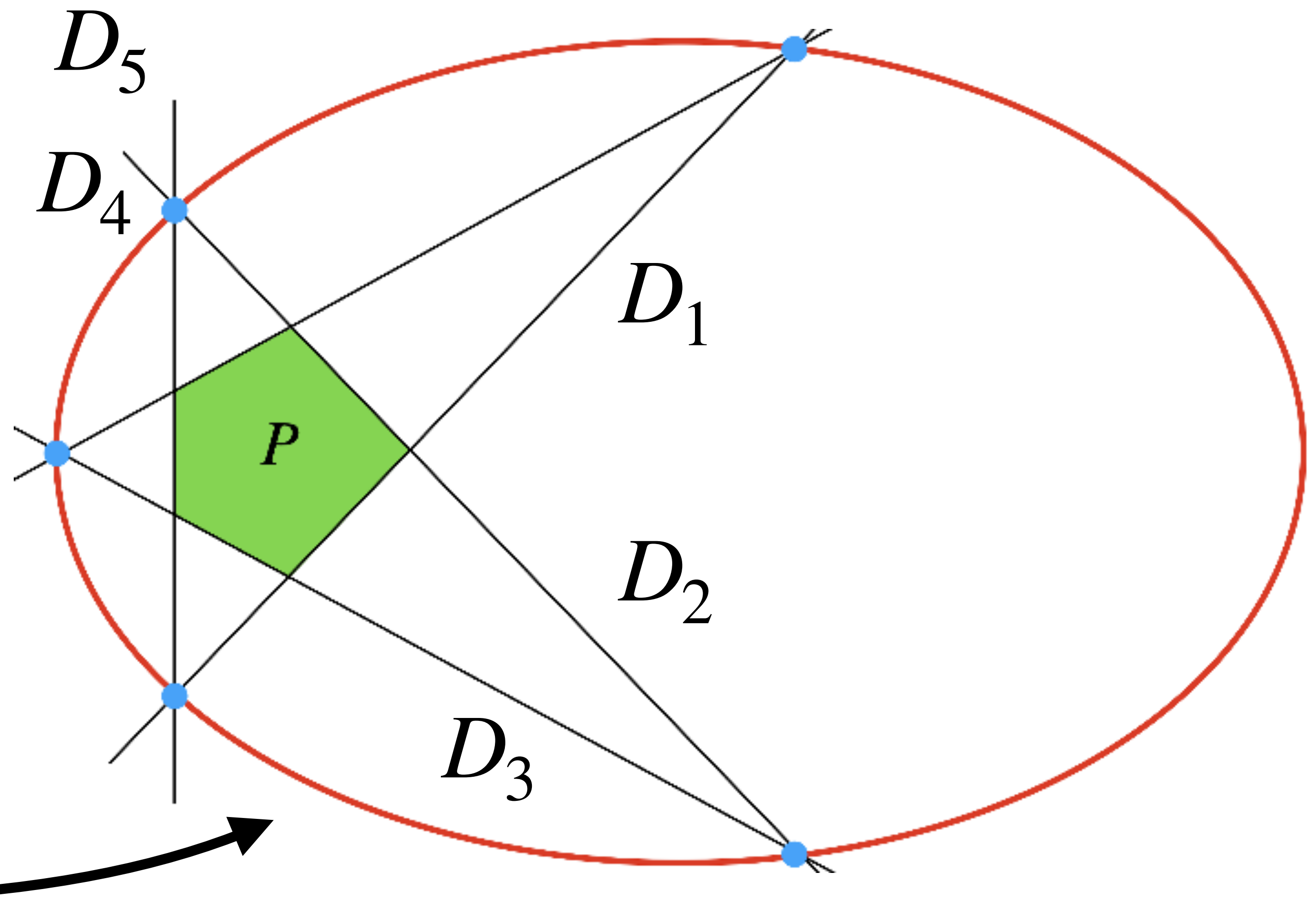
Proves previous conjectures: Amplitudes given by sum of volumes of positive Grassmannian cells. The tiles have cluster algebra properties.

[Even-Zohar, Lagre, Parisi, TEssler, Sherman-Bennett, Williams, arXiv:2310.17727]

Loop integrands from adjoint curves

The Amplituhedron gives a novel definition of the loop integrand, without reference to Feynman diagrams. [Arkani-Hamed, Trnka, 2013]

Triangulating the Amplituhedron is an open problem. *New idea*: fix the form from its zeros, which ensure absence of spurious singularities. The numerator corresponds to the *adjoint curve*.

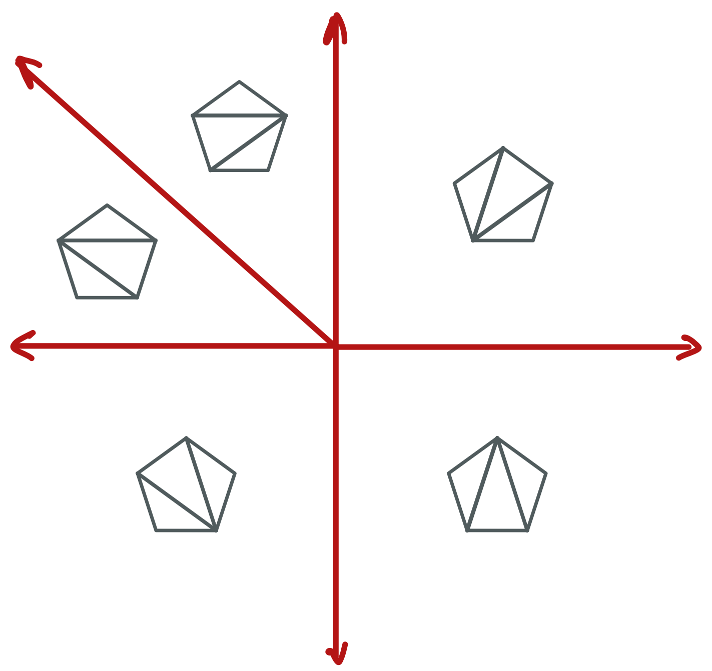


$$\omega \sim \frac{N}{D_1 D_2 D_3 D_4 D_5}$$

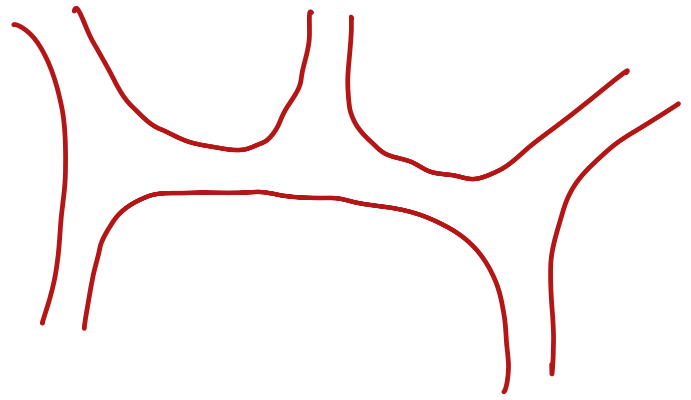
[Ranestad, Sinn, Telen, 2024;
Dian, Elia Mazzucchelli, Tellander, 2024]

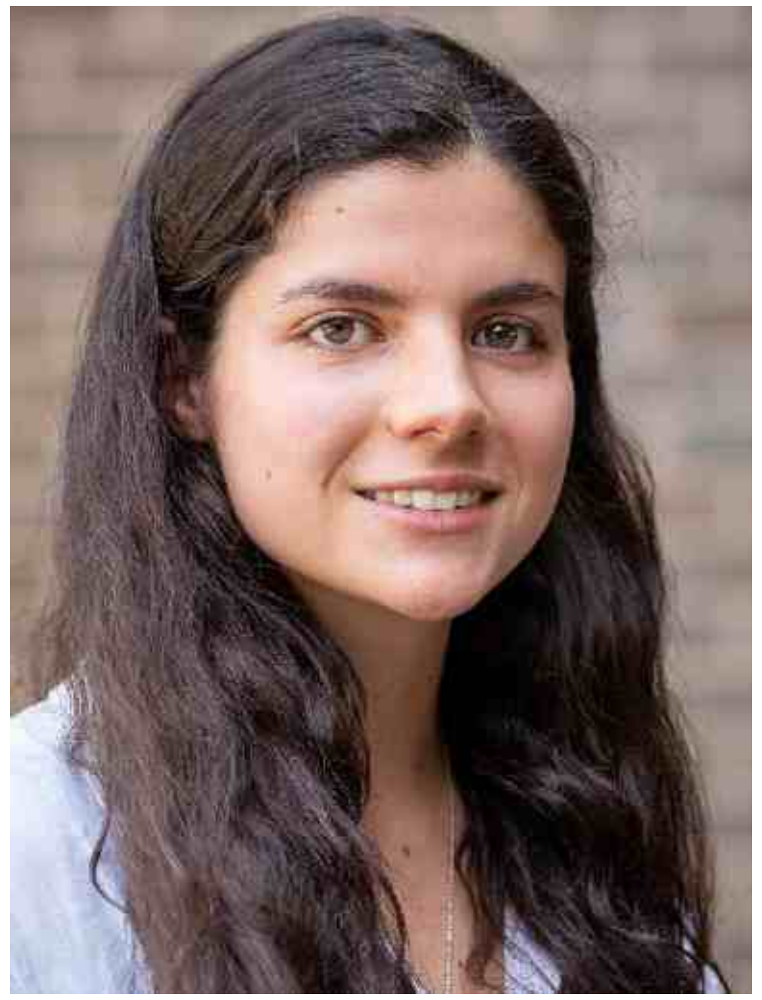
Global Feynman Representation for Scattering Amplitudes

How to build your favourite
Colored amplitudes tropically ?



(based on
arxiv 2402.06719)





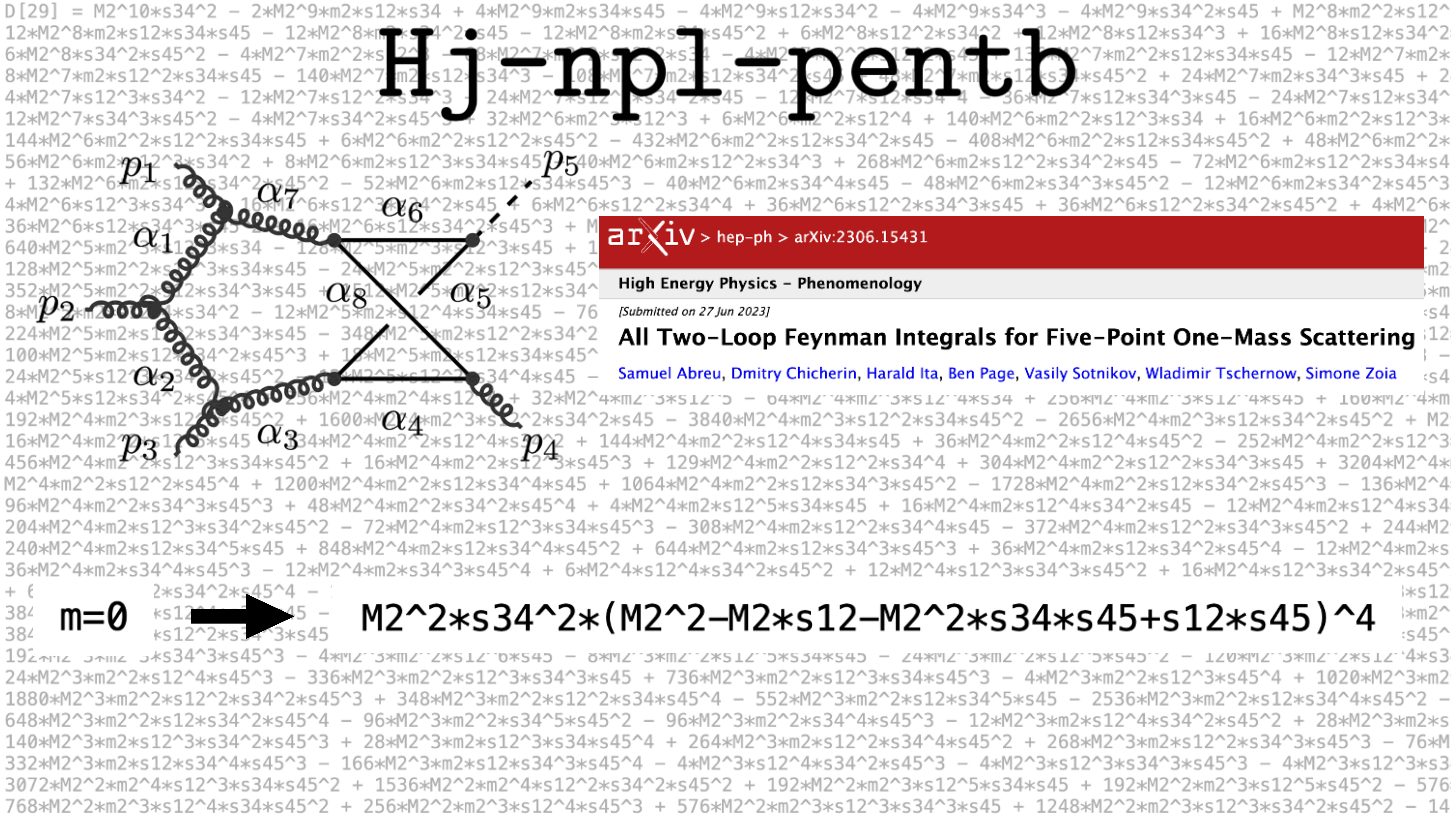
Carolina Figueiredo

February 2024
MPI, Leipzig

w/ N. Arkani-Hamed
H. Frost
G. Salvatori.

[from Carolina Figueiredo (Princeton)'s talk at Leipzig, 02/2024]

Landau Singularities Revisited: Computational Algebraic Geometry for Feynman Integrals

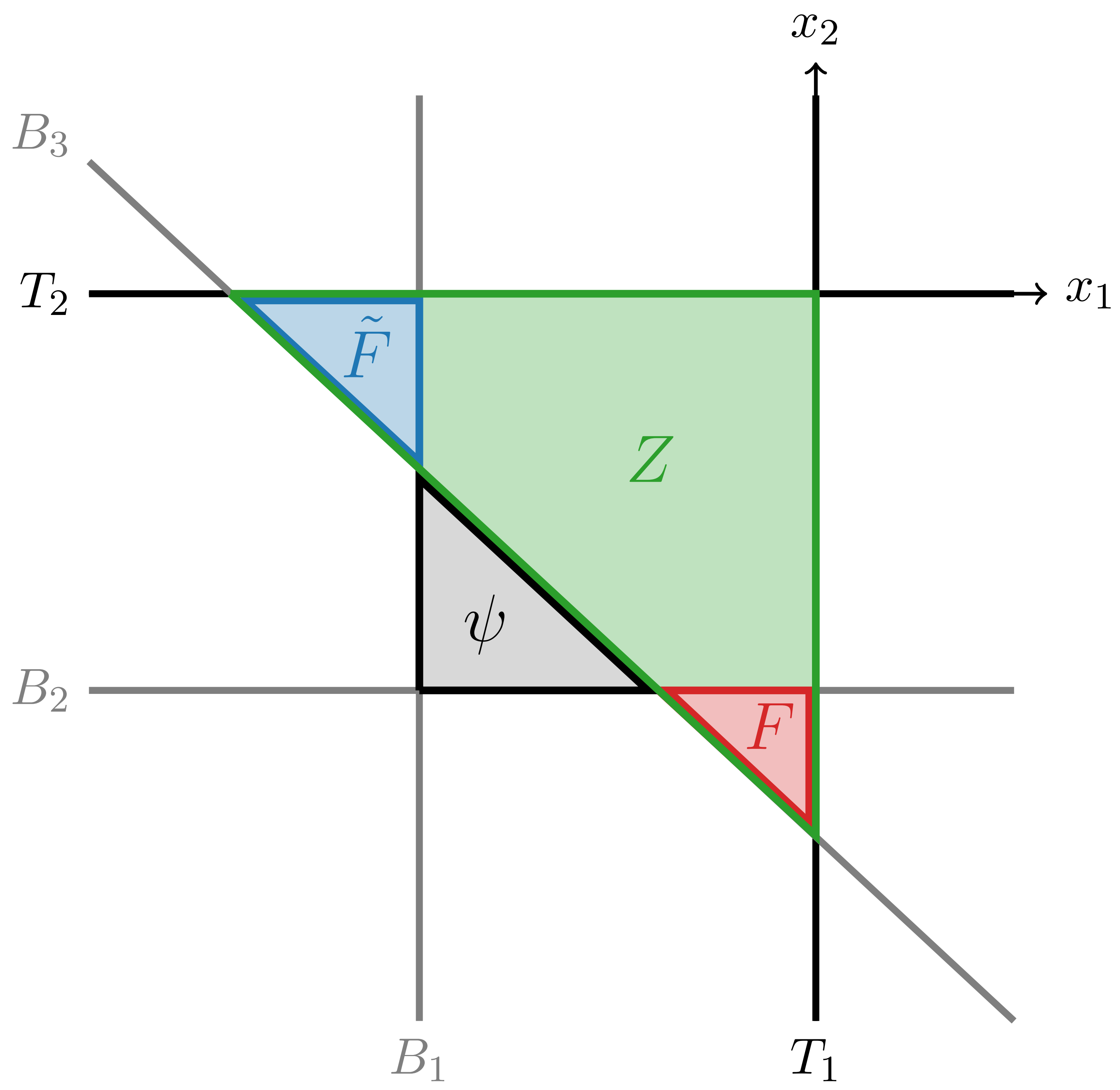


[from Claudia Fevola (INRIA, France)' talk at Leipzig, 02/2024]

[C. Fevola, S. Telen, S. Mizera, *Phys.Rev.Lett.* 132 (2024) 10, 101601]

Differential equations for cosmological correlators

$$\psi_{\mathrm{FRW}} \propto \int_0^\infty \mathrm{d}x_1 \int_0^\infty \mathrm{d}x_2 \, \frac{(x_1 x_2)^\varepsilon}{(X_1+X_2+x_1+x_2)(X_1+x_1+Y)(X_2+x_2+Y)} \, .$$

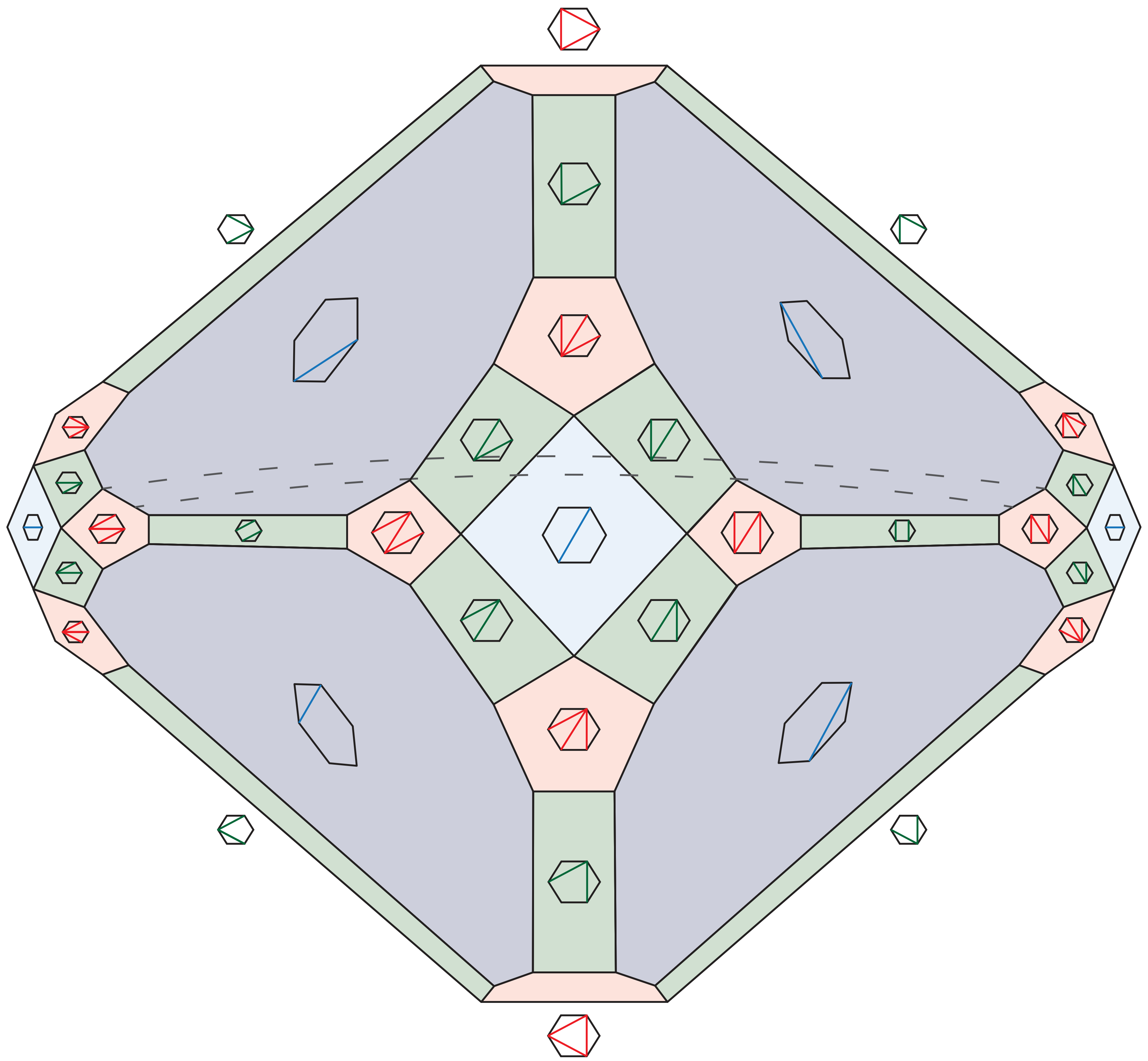


Differential equations for cosmological correlators determines from graphical rules.

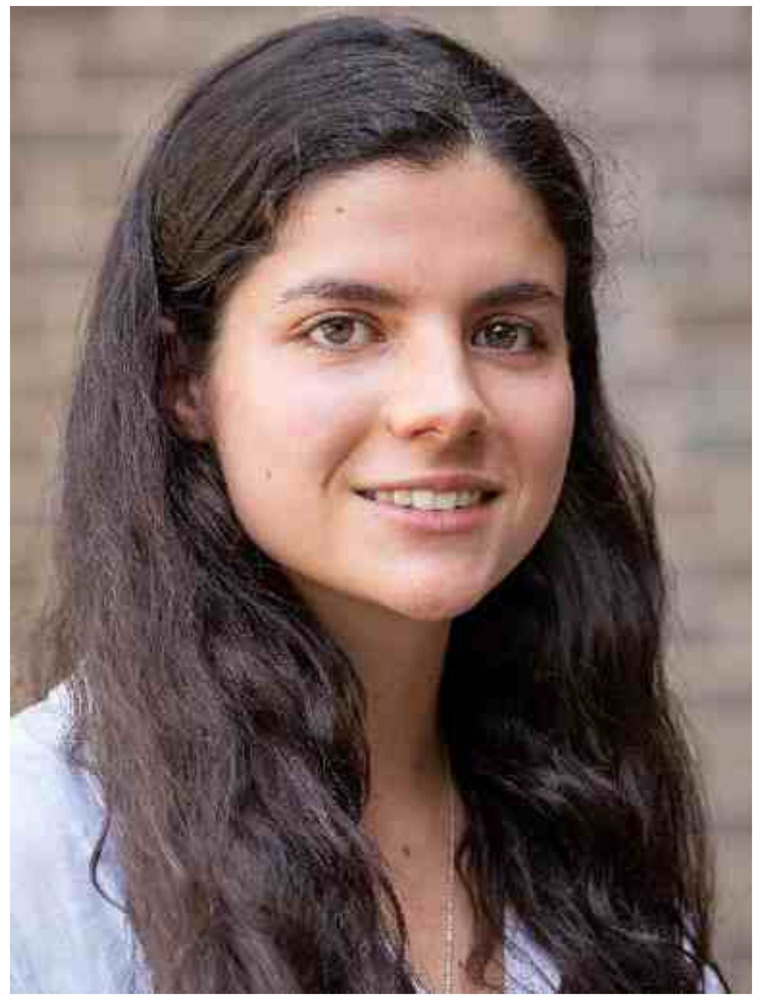
[Arkani-Hamed, Baumann, Hillmann, Joyce, Lee, Pimentel, 2023]

Discovery of Cosmohedra

Very recently, Nima Arkani-Hamed, Carolina Figueiredo, and Francisco Vazão, discovered a geometry for describing cosmological correlators, generalizing previous cosmological polytopes for individual graphs.



[Arkani-Hamed, Figueiredo, Vazão, arXiv: 2412.19881 [hep-th]]



Many opportunities to get involved

- Online bi-weekly seminar series (Wednesdays at 4 pm CET)
- June 29 - July 4, 2025: *‚The Amplituhedron: Structure, Combinatorics, and Positive Geometry‘*, SwissMAP Research Station in Les Diablerets
- February 17 - March 27, 2026: *‚Amplitudes and Algebraic Geometry‘* at the Erwin Schrödinger Institute, Vienna
- March 30 - April 10, 2026: Les Houches School of Physics

Stay tuned for more announcements!

Thank you!

henn@mpp.mpg.de
www.positive-geometry.com

universe+ is a cooperation of

MAX PLANCK INSTITUTE FOR PHYSICS

IAS INSTITUTE FOR ADVANCED STUDY

MAX PLANCK INSTITUTE FOR MATHEMATICS IN THE SCIENCES

UNIVERSITY OF AMSTERDAM

Funded by the European Union

European Research Council
Established by the European Commission